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ABSTRACT

This dissertation is concerned with the application of linearization techniques to the study of the response of non-linear dynamical systems subjected to periodic and random excitations.

A general method for generating an approximate solution of a multi-degree-of-freedom non-linear dynamical system is presented. This method relies on solving an optimum equivalent linear substitute of the original system.

The applicability of the method for determination of the amplitudes and phases of the approximate steady-state solution of a multi-degree-of-freedom non-linear system under harmonic monofrequency excitation is considered. The implementation of the method for several special classes of non-linear functions is discussed in detail. In addition, the manner in which the method may be applied to generate an approximate solution for the covariance matrix of the stationary random response of a multi-degree-of-freedom dynamical system subjected to stationary Gaussian excitation is outlined.

The potential of the method to treat transient solutions of non-linear systems is indicated in the context of the non-stationary response of a lightly damped and weakly non-linear oscillator subjected to monofrequency harmonic or to a Gaussian white noise disturbance. For both classes of excitation the method produces a first-order differential equation governing the response amplitude. The results pertinent to the harmonically excited oscillator are

compared with existing solutions. A non-stationary solution of the Fokker-Planck equation associated with the stochastic differential equation governing the response amplitude of the randomly excited oscillator is accomplished by perturbation techniques; the stationary solution is determined without making any approximation in the Fokker-Planck equation.

The new method for transient response is applied to the random response of a Duffing Oscillator and a Hysteretic System. The solution for the Duffing Oscillator is compared with data obtained by a Monte Carlo study.

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I. Introduction

The area of non-linear dynamics has become an interesting and challenging scientific branch for the engineer and the applied scientist. One major reason for this interest is the fact that it is not always possible to disregard non-linearities in today's complicated engineering systems and structures. In situations where a more complete understanding of the qualitative and the quantitative behavior of systems is required, it is often necessary to include non-linear effects. For example, in order to describe the response of structures to damaging excitations like earthquakes, consideration must be given to non-linear structural behavior. An excellent report of available techniques of non-linear analysis oriented towards earthquake engineering applications has been given by Iwan [1].

Typically, the response of a discrete non-linear dynamical system is described by a set of second order ordinary differential equations. Although non-linear second order differential equations have occupied the mathematician for some time, the techniques for exact closed form solution are quite limited. Existence and unicity of solutions can be demonstrated under certain conditions, but for only an admittedly small number of non-linear equations are exact solutions known. A good examination of the existence and uniqueness problem for deterministic non-linear differential equations is presented in reference [2].

In recent years considerable interest has arisen for problems of non-linear vibrations induced by random excitations. One major

incentive for such studies has been the problems which arise in the fields of aeronautical and space engineering. Another problem area which has also motivated research in the field of random vibrations is that of structural response to earthquakes. A common characteristic of such problems is that the excitation is usually so complex that it can only be described statistically. In addition, most physical systems respond linearly only for a limited range of external disturbances. Therefore, since under random excitation large responses may reasonably occur, a realistic study of system behavior under random disturbance must include some sort of non-linear analysis.

Usually, the response of a discrete non-linear dynamical system subjected to random excitation is modeled by a set of second order stochastic differential equations. It is clear that in this case the response of the non-linear system can no longer be considered as a single deterministic function, but should more properly be regarded as a family of functions characterized by some suitable statistics.

The theory of the response of linear systems to random disturbances is quite well developed and is available in common reference books [3, 4, 5]. However, the response of non-linear systems to random excitation is a broader and more complex topic. Recently, an excellent state of art report of the theory of non-linear random vibrations was presented by Caughey [6]. In this reference, among other topics, the existence and unicity of solutions of stochastic differential equations and the existence

and unicity of solutions of the associated Fokker-Planck equations are discussed.

Solution of the complete Fokker-Planck equation appropriate for a non-linear system provides the exact transition probability distribution function of the response to a Gaussian white noise process [7]. From the transition probability density function, the ordinary probability distribution of the response may be determined. Unfortunately, to date there is no systematic way of solving the complete Fokker-Planck equation for the general second-order non-linear system.

The stationary probability distribution function for the response of a non-linear system subjected to Gaussian white noise can be found by solving the stationary part of the corresponding Fokker-Planck equation. The broadest class of non-linear second order systems for which such a solution technique can be employed has been given by Caughey [8]. This class includes systems with non-hysteretic non-linear stiffness and non-linear damping which is a function of the amount of the energy of the system. This class of non-linear dynamical systems comprises all the particular problems for which exact solutions for the stationary probability density function have been obtained to date. Using this general solution, exact solutions may be found for a variety of systems including systems with cubic [9], tangent [10], and arc tangent [9] stiffness. Unfortunately, the above class of systems does not contain the interesting class of hysteretic systems. The latter

class of systems is distinctly important in modeling structural behavior during earthquakes.

The scarcity of exact solutions has necessitated the development of approximate analysis techniques for determining the response of non-linear systems subjected either to deterministic or to random excitations.

For the case of a single-degree-of-freedom non-linear oscillator subjected to deterministic excitations, most standard approximate techniques involve assuming a certain solution form containing some undetermined parameters. These parameters may be specified by minimizing the error residual obtained by substituting the presumed solution in the original non-linear dynamical system. Representative methods of this class are: the Poincaré-Linstend perturbation technique [11], Krylov-Bogoliubov-Mitropolsky asymptotic methods [12], and Galerkin's technique [13]. One significant restriction on most of the above methods is that the first-order approximation that they furnish is reliable only for weakly non-linear dynamical systems. More accurate solutions can be obtained by including more terms in the approximation, but the computational effort required usually makes this unprofitable.

For a single-degree-of-freedom non-linear oscillator subjected to harmonic excitation, the classical methods of equivalent linearization, harmonic balance, and energy balance may be used to generate an approximate first-order approximate solution [14]. Another approximate technique which produces the same first-order steady-state response equations as the methods of equivalent linearization,

harmonic balance, and energy balance is the method of slowly varying parameters [14]. The latter method may also be used to produce an approximate solution for the transient response of a non-linear harmonic oscillator subjected to a harmonic disturbance [15]. The interesting common feature of these "engineering" oriented mathematical techniques is that they can be applied to hysteretic systems as well [16,17].

Naturally, considerable effort has been devoted to adapting the above discussed approximate techniques of the deterministic theory to the field of random vibrations for single-degree-of-freedom non-linear oscillators. Very complete coverage of the currently available techniques for non-linear random vibration analysis may be found in the excellent review studies by Caughey [6] and Iwan [1]. One significant advance was the application of the classical perturbation method by Crandall [18] to random vibration analysis of single-degree-of-freedom oscillators with small non-linearity. However, major difficulties arise in the application of the perturbation method in the absence of linear viscous damping, or when the non-linear oscillator exhibits hysteretic behavior.

The adaptation of the classical equivalent linearization technique of the deterministic theory to systems subjected to random excitations was independently presented by Booton [19] and Caughey [20]. The principle of the method will be discussed herein during the general formulation given in subsequent chapters.

In addition to the techniques discussed above which are directly applicable to the stochastic differential equation, other

techniques have been devised for an approximate solution of the associated Fokker-Planck equation. These techniques [21] include separation of variables and eigenfunction expansions for the resulting eigenvalue problem, iterative solutions, numerical solutions, and a variety of other special techniques not of general use.

Undoubtedly, multi-degree-of-freedom non-linear systems are more difficult to treat analytically than their single-degree-of-freedom counterparts. For the class of problems pertinent to the steady-state response of harmonically excited non-conservative multi-degree-of-freedom non-linear systems there are essentially no exact solutions available. As a result, any detailed dynamic analysis of such a system must be accomplished using numerical integration of the equation of motion. However, this approach is generally very expensive. This fact reinforces the need for development of techniques for approximate analysis.

The same need exists for multi-degree-of-freedom non-linear systems subjected to random excitation. However, exact solutions for the stationary multi-dimensional probability function may be obtained for a certain class of problems. This must be done by solving the corresponding Fokker-Planck equation. Unfortunately, for the application of this method several requirements seldom met in physical systems must be satisfied [7].

Versions of the perturbation techniques, previously discussed in the context of a single-degree-of-freedom non-linear oscillator subjected to deterministic or random excitation, have also been applied to multi-degree-of-freedom non-linear systems [12, 22] but with rather limited success.

Caughey [20], Foster [23], Iwan and Yang [9], and Atalic and Utku [24] have extended the method of the equivalent linearization technique to encompass approximate solutions of the stationary random response of multi-degree-of-freedom non-linear oscillators.

Recently Iwan [25] extended the method of equivalent linearization to apply to multi-degree-of-freedom non-linear systems subjected to general excitations. However, several questions are raised by this development pertaining to the existence, uniqueness, and actual mechanization of the method for a specific class of excitations. These questions partially motivated the present investigation. A second incentive for this dissertation has been the formulation and application of the technique of equivalent linearization for the examination of the transient response of non-linear systems subjected to deterministic and random excitations.

In Chapter II the formulation of the generalized equivalent linearization method given in reference [25] is followed. However, the techniques used for the examination of such issues as the existence, uniqueness, and actual determination of an equivalent linear system are different from those used in reference [25]. The analysis performed in Chapter II serves to answer some of the questions raised by the development presented in reference [25]. In addition, a basis for using the equivalent linearization technique in the approximate analysis of transient solutions of non-linear dynamical systems is introduced.

In Chapter III the applicability of the method presented in Chapter II to the examination of the approximate steady-state

monofrequency harmonic or stationary Gaussian response of a multi-degree-of-freedom non-linear dynamical system is considered in detail. In section 3.1 a general introduction to the problem is given. In section 3.2 the existence and the uniqueness of an equivalent linear system for a multi-degree-of-freedom non-linear dynamical system is examined extensively. The principle of symmetric and non-symmetric non-linear functions is introduced. Procedures are given for the direct determination of an equivalent linear system for a class of non-linear systems frequently encountered in engineering applications. The mechanization of the method for the determination of the identification parameters of an approximate steady-state solution is discussed.

Section 3.3 contains a review of the manner in which the generalized method of equivalent linearization has been applied to generate an approximate solution for the covariance matrix of the stationary response of a multi-degree-of-freedom non-linear system subjected to random Gaussian excitation. This review is included in this thesis mainly for the purpose of completeness of the presentation. However, in addition, certain subtle points pertaining to the existence and uniqueness of an equivalent linear system are clarified.

In Chapter IV the method of equivalent linearization is applied to produce an approximate solution for the transient response of a non-linear dynamical system. In the present thesis, the method is applied only to single-degree-of-freedom non-linear dynamical systems.

In section 4.1 a general introduction to the problem of determining the response amplitude of a lightly damped and weakly non-linear oscillator is presented. In section 4.2 the case of harmonic monofrequency excitation is discussed. Specifically, an approximate first-order differential equation governing the response amplitude is obtained by energy considerations for the equivalent linear system associated with the harmonically excited non-linear oscillator. This equation is compared with the corresponding equation for the amplitude derived by standard perturbation techniques.

In section 4.3 the problem of determining the statistics of the amplitude of the response of the non-linear oscillator to a Gaussian white noise excitation is considered. Although extensive research has been conducted towards the study of the response of a randomly excited non-linear oscillator [6], it appears that only very limited work has been directed toward the investigation of the amplitude of the response [26,27]. The concept of the amplitude of the response, strictly speaking, has significance only when applied to the study of lightly damped weakly non-linear systems [26,28,29]. However, this is not a major limitation of the present method since such systems are frequently encountered in engineering applications.

For the application of the present method to a specific problem, the associated equivalent linear system must first be determined. Subsequently, a first-order stochastic differential equation governing the response amplitude may be obtained. In

the present thesis the equation for the amplitude is interpreted in the context of energy considerations for the equivalent linear system. An approximate non-stationary solution is generated for the Fokker-Planck equation associated with the equation governing the amplitude. For this purpose the technique of eigenfunctions expansion is used. In addition, the exact stationary solution is determined by a standard formula [30].

In Chapter V two example studies are conducted. These examples are thought to be representative of the manner in which the method developed in section 4.3 may be applied to specific problems.

The response of a lightly damped Duffing oscillator to a Gaussian white process is discussed in section 5.1. The stationary probability density function of the response of this system is readily available [9]. However, the problem of determining the non-stationary solution appears to be amenable only to very special [31] or Monte Carlo methods. Furthermore, it appears that presently there is no exact or approximate method for the determination of the statistics of the non-stationary response amplitude. In section 5.2 the time dependent mean value and standard deviation of the response amplitude are considered for several values of the non-linearity parameter of the problem. The stationary values of the statistics in discussion are examined even for a Duffing oscillator with severe non-linearity.

The response of a viscously damped hysteretic system to a Gaussian white process is discussed in section 5.2. For this type

of system several approximate analytical methods based on the principle of energy balance [32], similar systems, step by step linearization [33], discretization of response variables [34], or special approaches [31,35] have been proposed. Also, analog and digital simulation studies have been reported [36,37]. However, it appears that very limited research has been directed toward the statistics of the non-stationary response amplitude [38]. As an illustration of the potential of the method presented in section 4.3, the time dependent mean value of the response amplitude is calculated for a specific hysteretic model [39].

II. Generalized Method of Equivalent Linearization

2.1 Introduction

In this Chapter a generalization of the concept of equivalent linearization will be presented. For the case of harmonic excitation the principle of the method was first introduced in reference [12]. Later, the method was extended to the field of stochastic differential equations in references [19] and [20]. Recently, a generalization of the method of equivalent linearization for multi-degree-of-freedom non-linear dynamical systems subjected to general excitation was presented in reference [25]. Herein the general formulation given in reference [25] will be summarized. Special attention will be given to some subtle points about the existence and unicity of the equivalent linear system. In addition, the range of applicability of the method will be broadened so as to encompass problems involving the transient solutions of non-linear dynamical systems.

2.2 Non-Linear System

In this section the response to external load of a non-linear multi-degree-of-freedom dynamical system will be considered. The mathematical equation which describes the response of the system is

$$M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} + \underline{f}(\underline{x}, \dot{\underline{x}}) = \underline{g}(t) \quad , \tag{2.1}$$

where a dot above a variable denotes differentiation with respect to the independent variable t. M, C and K are constant $n \times n$ matrices, $\underline{f}(\underline{x}, \dot{\underline{x}})$ is an n-vector function of the dependent variable $\underline{x}$ and its

derivative, and $\underline{g}(t)$ is an n-vector function of the independent variable t.

Because of the scarcity of exact solutions of equation (2.1) when the function $\underline{f}(\underline{x}, \dot{\underline{x}})$ is non-linear, attention has been directed toward techniques of approximate analysis. For n small and for certain restrictions on $\underline{g}(t)$ and $\underline{f}(\underline{x}, \dot{\underline{x}})$, a number of standard analytic techniques can be adapted to the problem of generating an approximate solution of equation (2.1). Among these are perturbation methods and the methods of energy balance and slowly varying parameters. However, for n large or for g(t) and/or $f(\underline{x}, \dot{\underline{x}})$ of a more general form, the standard techniques often break down and are at best quite difficult to apply. Herein a systematic and easily mechanized method for generating an approximate solution of equation (2.1) will be presented.

The principle of the method is replacement of the non-linear dynamical system (2.1) by another auxiliary system for which the exact analytic formula for the solution is known. The replacement is made so as to be optimum with respect to some measure of the difference between the original and the auxiliary system. The auxiliary system need not necessarily be linear. In fact, for a single-degree-of-freedom non-linear oscillator references [40] and [41] use non-linear auxiliary systems. However, for the case of multi-degree-of-freedom systems only the response of linear systems is readily available. Therefore, it is realistic to deal only with the case of linear auxiliary system. Hereafter, this system will be called equivalent linear system.

2.3 Equivalent Linear System

2.3.1 Formulation

The equivalent linear system of (2.1) is defined as

$$M\ddot{\underline{x}} + (C + C_e)\dot{\underline{x}} + (K + K_e)\underline{x} = \underline{g}(t) \quad , \tag{2.2}$$

where C_e and K_e are t independent matrices. The matrices C_e and K_e are to be determined so that the difference $\underline{d}$ between the actual and the equivalent linear system is minimized for every $\underline{x}$ belonging to a certain class of functions of the independent variable t. The difference $\underline{d}$ may be written as

$$\begin{aligned} \underline{d} &= M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} + \underline{f}(\underline{x},\dot{\underline{x}}) - M\ddot{\underline{x}} - (C + C_e)\dot{\underline{x}} - (K + K_e)\underline{x} \\ &= \underline{f}(\underline{x},\dot{\underline{x}}) - C_e\dot{\underline{x}} - K_e\underline{x} \quad . \end{aligned} \tag{2.3}$$

According to this formulation the matrices C_e and K_e do depend on $\underline{x}$; therefore, if C_e and K_e are determined for that $x = \underline{x}_{LINEAR}$ which is the solution of the linear equation (2.2), it is reasonable to expect that this solution will be a fairly good approximate solution of the non-linear problem (2.1) as well. Since $\underline{x}_{LINEAR}$ clearly depends on C_e and K_e, a cyclic relation is established between C_e, K_e and $\underline{x}_{LINEAR}$. Utilizing this cyclic scheme the solution $\underline{x}_{LINEAR}$ can be determined. At this stage it is evident that the smaller the non-linear force $\underline{f}(\underline{x},\dot{\underline{x}})$ is, the more an equivalent linear system is suitable for the description of the response of the non-linear system (2.1). The smallness of $\underline{f}(\underline{x},\dot{\underline{x}})$ is usually denoted by including a small coefficient ϵ in front of

$\underline{f}(\underline{x}, \dot{\underline{x}})$ in equation (2.1). However, the present formulation will be carried out independently of the smallness of $\underline{f}(\underline{x}, \dot{\underline{x}})$.

2.3.2 Minimization Procedure

In order to determine the matrices C_e and K_e of the equivalent linear system it is necessary to establish a criterion for the minimization of the difference $\underline{d}$. Among the possible minimization criteria which include the minimization of the maximum value, the mean value, and the mean square value of $\underline{d}$, the last criterion is the most easily applied. For the case of a single-degree-of-freedom nonlinear oscillator, reference [42] indicates that there is no accuracy significant superiority of any of the above criteria over the others. Herein the Euclidean norm $||\underline{d}||_2$ of the difference $\underline{d}$ will be used as a measure of $\underline{d}$. The norm $||\underline{d}||_2$ is defined as

$$||\underline{d}||_2^2 = \underline{d}^T\underline{d} \ , \tag{2.4}$$

where $\underline{d}^T$ denotes the transpose of the vector $\underline{d}$. The minimization of $\underline{d}$ is performed according to the criterion

$$G(\underline{d}^T\underline{d}) = \text{minimum} \quad \forall \ \underline{x}(t) \ , \tag{2.5}$$

where $\underline{x}(t)$ belongs to the class of functions of t which are solutions of equation (2.1). In equation (2.5) G denotes an averaging operator which possesses several properties which assure certain characteristics of the equivalent linear system.

2.3.3 Properties of the Averaging Operator $\mathcal{G}$

The operator $\mathcal{G}$ is assumed to possess the following properties.

Property 1:

$$\frac{d}{dt}\,\mathcal{G}[x(t)] = 0$$

Property 2:

$$\mathcal{G}[x(t) + y(t)] = \mathcal{G}[x(t)] + \mathcal{G}[y(t)]$$

Property 3:

$$\mathcal{G}[x^2(t)] > 0 \quad ; \quad \forall\ x(t) \neq 0$$

$$\mathcal{G}[0] = 0$$

The operation of $\mathcal{G}$ on an $n \times n$ matrix Z is defined as

$$\mathcal{G}[Z] = \begin{pmatrix} \mathcal{G}(z_{11}), \ldots, \mathcal{G}(z_{1n}) \\ \cdots\cdots\cdots \\ \mathcal{G}(z_{n1}), \ldots, \mathcal{G}(z_{nn}) \end{pmatrix} , \qquad (2.6)$$

where

$$Z = \begin{pmatrix} z_{11}, \ldots, z_{1n} \\ \cdots\cdots \\ z_{n1}, \ldots, z_{nn} \end{pmatrix} .$$

2.3.4 Equations for the Equivalent Linear System Parameters

The necessary conditions for (2.5) to be true are

$$\frac{\partial}{\partial c_{ij}^e} \mathcal{G}(\underline{d}^T \underline{d}) = 0 \tag{2.7}$$

and $\forall\ i,j;\quad i,j = 1,\ldots,n$

$$\frac{\partial}{\partial k_{ij}^e} \mathcal{G}(\underline{d}^T d) = 0 \ , \tag{2.8}$$

where c_{ij}^e and k_{ij}^e are the (i, j) elements of the matrices C_e and K_e respectively. Equation (2.5) can be rewritten as

$$\mathcal{G}[d_1^2 + \cdots + d_n^2] = \text{minimum} \ , \tag{2.9}$$

where

$$\underline{d} = (d_1, \ldots, d_n)^T \ . \tag{2.10}$$

Upon using the linearity property of $\mathcal{G}$, equation (2.9) can be put in the form

$$\sum_{i=1}^{n} D_i^2 = \text{minimum} \ , \tag{2.11}$$

where D_i is defined by

$$D_i^2 = \mathcal{G}(d_i^2) \ ; \qquad i = 1,\ldots,n \ . \tag{2.12}$$

Because of equation (2.3), equation (2.12) becomes

$$D_i^2 = G[f_i - \sum_{j=1}^{n} (c_{ij}^e \dot{x}_j + k_{ij}^e x_j)]^2 \quad ; \qquad i = 1, \ldots, n \tag{2.13}$$

where

$$\underline{f} = (f_1, \ldots, f_n)^T \quad . \tag{2.14}$$

Examining equation (2.13) it is seen that D_i depends only on c_{ij}^e and k_{ij}^e, where $j = 1, \ldots, n$. Therefore, the minimization criterion (2.9) can be expressed as

$$D_i^2 = \text{minimum} \quad ; \qquad i = 1, \ldots, n \quad . \tag{2.15}$$

The necessary conditions for (2.15) to be true are

$$\frac{\partial}{\partial c_{ij}^e} (D_i^2) = 0 \quad ; \qquad j = 1, \ldots, n \tag{2.16}$$

and

$$\frac{\partial}{\partial k_{ij}^e} (D_i^2) = 0 \quad ; \qquad j = 1, \ldots, n \quad . \tag{2.17}$$

Expanding equations (2.16) and (2.17) and utilizing equation (2.14) gives

$$G(\dot{x}_j f_i) = \sum_{s=1}^{n} [c_{is}^e G(\dot{x}_s \dot{x}_j) + k_{is}^e G(x_s \dot{x}_j)] \tag{2.18}$$

and

$$G(x_j f_i) = \sum_{s=1}^{n} [c_{is}^e G(\dot{x}_s x_j) + k_{is}^e G(x_s x_j)] \quad . \tag{2.19}$$

Using definition (2.6), equations (2.18) and (2.19) can be rewritten in the compacted form

$$G[f_i \hat{\underline{x}}] = G[\hat{\underline{x}}\,\hat{\underline{x}}^T] \begin{bmatrix} k_{i*}^{e\,T} \\ c_{i*}^{e\,T} \end{bmatrix} ; \qquad i = 1, \ldots, n \tag{2.20}$$

where

$$\hat{\underline{x}} = \begin{pmatrix} \underline{x} \\ \dot{\underline{x}} \end{pmatrix} \tag{2.21}$$

and k_{i*}^e , c_{i*}^e are the ith rows of the matrices K_e and C_e respectively.

2.3.5 Examination of the Minimum

From equation (2.13) it is recognized that the quantity D_i^2 is simply a quadratic polynomial of the parameters c_{ij}^e and k_{ij}^e. Therefore, its mixed partial derivatives with respect to c_{ij}^e and k_{ij}^e of order higher than 2 vanish. Hence, if the value of D_i^2 which corresponds to another set of parameters

$$c_{ij}^{\prime e} = c_{ij}^e + \Delta c_{ij}^e \tag{2.22}$$

$$k_{ij}^{\prime e} = k_{ij}^e + \Delta k_{ij}^e \tag{2.23}$$

is considered, the following Taylor's expansion around c_{ij}^e and k_{ij}^e can be made.

$$D_i^2(c_{ij}'^e, k_{ij}'^e) = D_i^2(c_{ij}^e, k_{ij}^e) + \sum_{j=1}^{n} \left(\frac{\partial D_i^2}{\partial c_{ij}^e} \Delta c_{ij}^e + \frac{\partial D_i^2}{\partial k_{ij}^e} \Delta k_{ij}^e \right)$$

$$j = 1, \ldots, n \qquad j = 1, \ldots, n$$

$$+ \tfrac{1}{2} \sum_{\rho, m=1}^{n} \frac{\partial^2 D_i^2}{\partial c_{i\rho}^e \partial c_{im}^e} \Delta c_{i\rho}^e \Delta c_{im}^e$$

$$+ \tfrac{1}{2} \sum_{\rho, m=1}^{n} \frac{\partial^2 D_i^2}{\partial k_{i\rho}^e \partial k_{im}^e} \Delta k_{i\rho}^e \Delta k_{im}^e$$

$$+ \tfrac{1}{2} \sum_{\rho, m=1}^{n} \frac{\partial^2 D_i^2}{\partial c_{i\rho}^e \partial k_{im}^e} \Delta c_{i\rho}^e \Delta k_{im}^e \quad . \tag{2.24}$$

Because of conditions (2.16) and (2.17) the first sum in equation (2.24) is zero. By considering once more equation (2.13), equation (2.24) can further be simplified as

$$D_i^2(c_{ij}'^e, k_{ij}'^e) = D_i^2(c_{ij}^e, k_{ij}^e) + G\left[\sum_{j=1}^{n} (\Delta c_{ij}^e \dot{x}_j + \Delta k_{ij}^e x_j) \right]^2 \quad . \tag{2.25}$$

$$j = 1, \ldots, n \qquad j = 1, \ldots, n$$

By property 3 of section 2.3.3,

$$G\left[\sum_{j=1}^{n} (\Delta c_{ij}^e \dot{x}_j + \Delta k_{ij}^e x_j) \right]^2 > 0 \; ; \qquad i = 1, \ldots, n$$

$$\text{for} \quad \sum_{j=1}^{n} (\Delta c_{ij}^e \dot{x}_j + \Delta k_{ij}^e x_j) \neq 0 \tag{2.26}$$

and

$$\mathcal{G}\left[\sum_{j=1}^{n}\left(\Delta c_{ij}^{e}\dot{x}_j+\Delta k_{ij}^{e}x_j\right)\right]^2 = 0 \quad ; \qquad i = 1,\ldots,n$$

$$\text{for} \quad \sum_{j=1}^{n}\left(\Delta c_{ij}^{e}\dot{x}_j+\Delta k_{ij}^{e}x_j\right) = 0 \quad . \tag{2.27}$$

Because of inequality (2.26), equation (2.25) yields

$$D_i^2(c_{ij}^{e\prime},k_{ij}^{e\prime}) \geq D_i^2(c_{ij}^{e},k_{ij}^{e}) \quad ; \qquad i = 1,\ldots,n \tag{2.28}$$

where the equality holds if and only if equation (2.27) is true. Inequality (2.28) assures that if a linear system exists with damping matrix C_e and stiffness matrix K_e satisfying condition (2.20), then the value of $\mathcal{G}(\underline{d}^T\underline{d})$ which corresponds to C_e and K_e is <u>not bigger</u> than the value of $\mathcal{G}(\underline{d}^T\underline{d})$ which corresponds to any other pair of damping and stiffness matrices.

Obviously, relation (2.27) corresponds to the case that the member $\hat{\underline{x}}$ of the class of possible solutions of the equivalent linear system (2.2) has linearly dependent components for every value of the independent variable t. Hence, if the components are linearly independent it can be assured that the value of $\mathcal{G}(\underline{d}^T\underline{d})$ which corresponds to C_e and K_e is an absolute minimum.

2.3.6 <u>Existence and Uniqueness of the Equivalent Linear System</u>

In this section it will be proved that the linear dependence of the components of $\hat{\underline{x}}$ is also important for the existence and the

uniqueness of the equivalent linear system.

Equation (2.20) must be solved so that the matrices C_e and K_e will be found in terms of $\hat{\underline{x}}$. It is easily recognized that equation (2.20) is simply a system of 2n equations linear in c_{ij}^e and k_{ij}^e. Therefore, a unique solution of (2.20) exists if and only if the matrix $\mathcal{G}[\hat{\underline{x}}\,\hat{\underline{x}}^T]$ is non-singular. If the matrix $\mathcal{G}[\hat{\underline{x}}\,\hat{\underline{x}}^T]$ is singular, a non-unique solution of (2.20) exists only for special types of non-linear functions $\underline{f}(\underline{x},\dot{\underline{x}})$. The following theorem provides a criterion for whether or not the matrix $\mathcal{G}[\hat{\underline{x}}\,\hat{\underline{x}}^T]$ is singular.

Theorem

Given an s-dimensional vector space V defined over the set of real numbers, the matrix $\mathcal{G}(\hat{\underline{x}}\,\hat{\underline{x}}^T)$ is singular if and only if $2n > s$, where

$$\hat{\underline{x}} = \begin{pmatrix} \underline{x} \\ \dot{\underline{x}} \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \quad x_i, \dot{x}_i \in V \; ; \quad i = 1,\ldots,n \quad .$$

Proof:

Sufficient Condition

It is assumed that

$$2n > s \quad . \tag{2.28a}$$

Therefore, the 2n elements of V are linearly dependent. Consequently, there exist 2n real numbers a_j, b_j; $j = 1,\ldots,n$, not all zero, such that

$$\sum_{j=1}^{n} (a_j x_j + b_j \dot{x}_j) = 0 \quad . \tag{2.29}$$

Using vector notation equation (2.29) can be rewritten as

$$\hat{x}^T . \underline{u} = \underline{0} \quad , \tag{2.30}$$

where

$$\underline{u}^T = (a_1, \ldots, a_n, b_1, \ldots, b_n) \neq \underline{0} \quad . \tag{2.31}$$

Multiplying equation (2.30) by $\hat{\underline{x}}$ from the left gives

$$\hat{\underline{x}} \hat{\underline{x}}^T \underline{u} = \underline{0} \quad . \tag{2.32}$$

Applying the operator $\mathcal{G}$ on both sides of (2.32) and taking into consideration the properties of $\mathcal{G}$ which were assumed in section 2.3.3 yields

$$\mathcal{G}(\hat{\underline{x}} \hat{\underline{x}}^T)\underline{u} = \underline{0} \quad . \tag{2.33}$$

Since $\underline{u} \neq \underline{0}$, the matrix $\mathcal{G}(\hat{\underline{x}} \hat{\underline{x}}^T)$ is singular.

Necessary Condition

It is assumed that the matrix $\mathcal{G}(\hat{\underline{x}} \hat{\underline{x}}^T)$ is singular. Hence, there exists a vector $\underline{u} \neq \underline{0}$ such that equation (2.33) is satisfied. Multiplying both sides of equation (2.33) by $\underline{u}^T$ yields

$$\underline{u}^T \mathcal{G}(\hat{\underline{x}} \hat{\underline{x}}^T) \underline{u} = \mathcal{G}(\underline{u}^T \hat{\underline{x}} \hat{\underline{x}}^T \underline{u}) = \mathcal{G}[(\hat{\underline{x}}^T \underline{u})^2] = 0 \quad . \tag{2.34}$$

Because of property 3 of the operator $\mathcal{G}$, relation (2.34) implies equation (2.30). Therefore, the 2n elements of V, $x_i, \dot{x}_i$; $i = 1, \ldots, n$

are linearly dependent. Hence, the number 2n must be bigger than the dimension s of the vector space V.

Considering the above theorem and the results of last section, it is seen that the necessary and sufficient condition for the existence and uniqueness of an equivalent linear system is the same as the condition which assures an absolute minimum of $G(\underline{d}^T\underline{d})$.

2.3.7 Conclusions about the Equivalent Linear System

A unique equivalent linear system (2.2) exists for the n-degree-of-freedom non-linear system (2.1) if and only if the condition $2n \leq s$ is satisfied. The symbol s stands for the dimension of the vector space of the approximate solutions of (2.1). If the above condition is satisfied, the value of the quantity $G(\underline{d}^T\underline{d})$ is minimized. The symbol $\underline{d}$ stands for the difference between the actual non-linear system (2.1) and the equivalent linear system (2.2). If $s > 2n$, the existence of an equivalent linear system is not assured. If such a system with matrices C_e and K_e exists, it is not unique. However, the value of $G(\underline{d}^T\underline{d})$ which corresponds to this system is not bigger than the value of $G(\underline{d}^T\underline{d})$ which corresponds to any other pair of matrices C_e and K_e. Therefore, it can still be considered as an optimum substitute of the non-linear system (2.1).

2.4 Mechanization of the Method

In this section a brief outline of the manner in which the presented method may be applied is given. The details for the specific categories of problems will be discussed in subsequent chapters.

Given the non-linear system (2.1), it is first necessary to specify an acceptable class of approximate solutions $\underline{x}$. The selection of the class of approximate solutions depends on the excitation $\underline{g}(t)$ and the system (2.1). If, for example, $\underline{g}(t)$ is a periodic mono-frequency vector force, it is plausible to assume that the steady-state periodic solution $\underline{x}$ of (2.1) will have the same frequency as the excitation $\underline{g}(t)$. Each member of the solution class must be identified by some parameters (amplitude and phase for example for the class of harmonic solution functions). For the class of accepted approximate solution functions the equivalent linear system (2.2) must be determined. For this purpose the averaging operator $\mathfrak{A}$ which is expected to be the most suitable for the specific problem is selected. Next the matrices C_e and K_e need to be determined by solving equation (2.20). Clearly the matrices C_e and K_e depend on the identification parameters of $\underline{x}$. The last step in this procedure is to solve the equivalent linear system (2.2) and derive equations for the identification parameters of $\hat{x}$ in terms of the matrices C_e and K_e. Evidently, the method provides equations, either algebraic or differential, for the determination of important parameters of the approximate solution of the non-linear system. To date the method of equivalent linearization has only been used to deduce algebraic equations for the parameters of the response.

In summary the mechanization of the method requires:

1. Identification of a class of approximate solution functions and the parameters defining each member of the class.
2. Selection of the norm of the difference vector $\underline{d}$.

3. Selection of the averaging operator $\mathcal{G}$.
4. Determination of the matrices C_e and K_e of the equivalent linear system in terms of the identification parameters of $\underline{x}$.
5. Solution of the equivalent linear system to obtain equations for the specification of the identification parameters of $\underline{x}$.

2.5 Accuracy of the Method

No analytical work towards the determination of the accuracy of the method will be done. This task appears to be formidable and deviates from the objectives of this thesis. For the pure mathematician a question remains open. In what sense does the minimization of the difference between the non-linear and the equivalent linear system imply minimization of the difference between the exact and the approximate solution of the non-linear system? In subsequent chapters the accuracy of the method will be taken up again during the examination of specific categories of problems. At that time a more detailed discussion of the matter will be presented.

2.6 Summary

A general method for derivation of approximate solutions of multi-degree-of-freedom non-linear dynamical systems has been presented. The principle of the method is to replace the non-linear system with an equivalent linear system in such a way that an average of the difference between the two systems is minimized. Next the solution of the non-linear system is approximated by the solution of the equivalent linear system. The minimization of an average Euclidean norm of the above mentioned difference is

adopted as the criterion for the determination of the equivalent linear system. The averaging operator is assumed to possess certain properties. Equations for the specification of the parameters of the equivalent linear system are derived and the existence and the uniqueness of a solution of these equations is examined. The existence of the absolute minimum of the averaged norm of the difference is examined as well. During these examinations several properties of the averaging operator are exploited. A systematic procedure for the mechanization of the method is given.

III. Application to Steady-State Response of Multi-Degree-of-Freedom Non-Linear Systems

3.1 Introduction

In this chapter the generalized method of equivalent linearization presented in Chapter II will be applied for the examination of the steady-state harmonic and the stationary Gaussian response of multi-degree-of-freedom non-linear dynamical systems. This class of problems is of importance for practical engineering situations. For example, equation (2.1) may be a fairly good model of a multi-story building structure, where the matrices M, C and K are respectively the mass, damping, and stiffness matrices of the structure. The non-linear function $\underline{f}(\underline{x}, \dot{\underline{x}})$ may represent the effect of geometrical or material non-linearities. Hysteresis is one example of the latter category of non-linearity. One of the advantages of the present method is its potentiality to overcome the mathematical complexities which are associated with hysteretic systems. It is obvious that the hysteretic response of a multi-story structure is not the only candidate for the application of the method. Other engineering systems like aircraft, missiles, offshore structures, ships, and so on often require a non-linear analysis as well.

Considering the stimulus $\underline{g}(t)$ on the right hand side of equation (2.1), interest is focused on the cases of harmonic monofrequency or stationary Gaussian excitation.

Interest in harmonic response arises frequently when rotating machinery is present or some filtering process reduces the excitation to a monofrequency vector. The identification of structural

parameters of multi-story buildings through forced vibration tests is another example for which the assumption of harmonic excitation is justified.

Because of the random nature of various physical processes as earthquakes, acoustical noise, and sea waves, a statistical analysis of their effects on various systems is generally indicated. Analysis of a large number of individual records of a random process permits determination of the statistical measures which can be expected to also characterize future records. If it is assumed that the best substitute for an actual record of a random process is the average of many similar records, the modeling of such a natural process by a Gaussian process would be in accordance with the central limit theorem. Hence, it is thought that the study of the response of multi-degree-of-freedom non-linear systems subjected to Gaussian excitation is of engineering importance.

3.2 Steady-State Harmonic Response

3.2.1 Specification of the Equivalent Linear System

Consider the non-linear system (2.1) for the special case in which

$$\underline{g}(t) = \mathrm{Re}(e^{i\omega t})\,\underline{G} \ , \tag{3.1}$$

where

$$\underline{G}^T = (g_1, \ldots, g_n) \tag{3.2}$$

and ω is the angular frequency of the excitation. The symbol Re denotes the real part of the complex number inside the parenthesis.

Because of equation (3.1) the non-linear system (2.1) may be rewritten as

$$M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} + \underline{f}(\underline{x},\dot{\underline{x}}) = \mathrm{Re}(e^{i\omega t})\underline{G} \ . \tag{3.3}$$

Presently, no systematic method is available for the derivation of the exact solution of equation (3.3). Any "exact" solutions must be found by numerical integration of an initial value problem. This technique of solution is very expensive if it is to be used to obtain a steady-state solution.

It is here assumed that equation (3.3) possesses a steady-state solution of the form

$$\underline{x(t)} = [x_1(t),\ldots,x_n(t)]^T \tag{3.4}$$

where

$$x_i(t) = \mathrm{Re}(X_i e^{i\omega t}) \ ; \qquad i = 1,\ldots,n \tag{3.5}$$

and X_i are complex numbers. Following the analysis given in section (2.3) the equivalent linear system is defined as

$$M\ddot{\underline{x}} + (C + C_e)\dot{\underline{x}} + (K + K_e)\underline{x} = \mathrm{Re}(e^{i\omega t})\underline{G} \ , \tag{3.6}$$

where C_e and K_e are solution dependent matrices such that the difference

$$\underline{d} = \underline{f}(\underline{x},\dot{\underline{x}}) - C_e\dot{\underline{x}} - K_e\underline{x} \tag{3.7}$$

is minimized. The Euclidean norm $||\underline{d}||_2$ of the difference $\underline{d}$ is adopted as a measure of the size of $\underline{d}$. The norm $||\underline{d}||_2$ is defined as

$$||\underline{d}||_2^2 = \underline{d}^T\underline{d} \quad . \tag{3.8}$$

The minimization of $\underline{d}$ is performed according to the criterion

$$\mathbb{G}(\underline{d}^T\underline{d}) = \text{minimum} \quad \forall\ \underline{x}(t) \quad , \tag{3.9}$$

where $\underline{x}(t)$ is of the form of equation (3.4). $\mathbb{G}$ is the averaging operator. Since it was assumed that the steady-state solution of equation (3.3) is periodic with period

$$T = \frac{2\pi}{\omega} \quad , \tag{3.10}$$

a logical averaging operator is the average over one period of the solution. That is

$$\mathbb{G}[x(t)] = \frac{1}{T}\int_0^T \underline{x}(t)dt \quad . \tag{3.11}$$

Clearly, $\mathbb{G}$ possesses all the properties discussed in (2.3.3). The operation of $\mathbb{G}$ on a square matrix is defined as in (2.6).

The necessary conditions for criterion (3.9) to be satisfied are

$$\frac{\partial}{\partial c_{ij}^e}\int_0^T (\underline{d}^T\underline{d})\,dt = 0 \tag{3.12}$$

and $\forall\ i,j\ ; \qquad i,j = 1,\ldots,n$

$$\frac{\partial}{\partial k_{ij}^e}\int_0^T (\underline{d}^T\underline{d})dt = 0 \quad , \tag{3.13}$$

where c_{ij}^e and k_{ij}^e are the (i, j) elements of the matrices C_e and K_e respectively. Following the analysis performed in (2.3.4) it is found that the matrices C_e and K_e must satisfy the equation

$$\int_0^T [f_i \hat{\underline{x}}] \, dt = \int_0^T [\hat{\underline{x}} \, \hat{\underline{x}}^T] \begin{bmatrix} k_{i*}^{e\,T} \\ c_{i*}^{e\,T} \end{bmatrix} ; \qquad i = 1, \ldots, n \qquad (3.14)$$

where

$$\hat{\underline{x}} = \begin{pmatrix} \underline{x} \\ \dot{\underline{x}} \end{pmatrix} . \qquad (3.15)$$

k_{i*}^e and c_{i*}^e are the ith rows of the matrices K_e and C_e respectively.

3.2.2 Examination of the Minimum

As indicated in parts (2.3.5) and (2.3.6) the dimension s of the vector space V of the approximate solution functions is important for the behavior of the quantity $\frac{1}{T} \int_0^T \underline{d}^T \underline{d} \, dt$. Since the components of the approximate steady-state solution have been assumed to be of the form (3.5), a basis for the vector space V is the set

$$U = \{ \sin \omega t, \cos \omega t \} . \qquad (3.16)$$

Therefore, the value of s for the case of harmonic response is

$$s = 2 . \qquad (3.17)$$

According to the general analysis performed in part (2.3.5) the following conclusions may be drawn for the case of harmonic response.

1. For a single-degree-of-freedom (n = 1) non-linear system, the value of the quantity $\frac{1}{T}\int_0^T \underline{d}^T\underline{d}\,dt$ corresponding to the equivalent linear system is minimized. This result has been found by other investigators as well [1].
2. For a multi-degree-of-freedom ($n \geq 2$) non-linear system, the value of the quantity $\frac{1}{T}\int_0^T \underline{d}^T\underline{d}\,dt$ corresponding to an equivalent linear system with matrices C_e and K_e is <u>not</u> bigger than the value of $\int_0^T \underline{d}^T\underline{d}\,dt$ corresponding to any other pair of matrices C_e and K_e.

3.2.3 <u>Existence and Uniqueness of the Equivalent Linear System</u>

It was seen in part (2.3.6) that the dimension s of the vector space V is also important in determining the existence and the uniqueness of the approximate solution. The value of s is given by equation (3.17). Utilizing the results of the general discussion presented in part (2.3.6), the following conclusions may be drawn for the case of harmonic response.

1. For a single-degree-of-freedom (n = 1) non-linear system, the equivalent linear system (3.6) exists and is unique. This result has been found by other investigators as well [1].

2. For a multi-degree-of-freedom ($n \geq 2$) non-linear system, an equivalent linear system (3.6) does not in general exist. This is due to the fact that the matrix $\int_0^T \underline{\hat{x}}\,\underline{\hat{x}}^T dt$ is singular for $n \geq 2$. If an equivalent linear system with matrices C_e and K_e does exist, this system will not be unique.

The next task in the present analysis will be the investigation of the existence of possible particular solutions of equation (3.14). Transposing both sides of equation (3.14) gives

$$\int_0^T [f_i \underline{\hat{x}}^T]\, dt = [k_{i*}^e, c_{i*}^e] \int_0^T \underline{\hat{x}}\,\underline{\hat{x}}^T dt \quad . \tag{3.18}$$

Using equation (3.4), the vector $\underline{x}$ and its derivative $\dot{\underline{x}}$ can be expressed as

$$\underline{x}^T = \mathrm{Re}[e^{i\omega t}(X_1, \ldots, X_n) \tag{3.19}$$

and

$$\dot{\underline{x}}^T = \mathrm{Re}[e^{i\omega t}(Y_1, \ldots, Y_n) \quad , \tag{3.20}$$

where

$$Y_i = i\omega X_i \quad ; \qquad i = 1, \ldots, n \quad . \tag{3.21}$$

Upon using equations (3.19) and (3.20), the vector $\hat{\underline{x}}$ can be written in the form

$$\hat{\underline{x}} = \mathrm{Re}[e^{i\omega t}\underline{S}] \quad , \tag{3.22}$$

where

$$\underline{S}^T = (X_1, \ldots, X_n, Y_1, \ldots, Y_n) \quad . \tag{3.23}$$

Next, consider the relation

$$\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2) \quad . \tag{3.24}$$

This relation is valid for any pair of complex numbers z_1 and z_2. The symbol Im stands for the imaginary part of the complex number inside the parenthesis. Using property (3.24), the expresssion $\int_0^T \hat{\underline{x}}\,\hat{\underline{x}}^T\,dt$ can be manipulated as follows

$$\int_0^T \hat{\underline{x}}\hat{\underline{x}}^T dt = \int_0^T \mathrm{Re}(e^{i\omega t}\underline{S})\mathrm{Re}(e^{i\omega t}\underline{S}^T)\,dt$$

$$= \int_0^T [\mathrm{Re}(e^{i\omega t})\mathrm{Re}(\underline{S}) - \mathrm{Im}(e^{i\omega t})\mathrm{Im}(\underline{S})][\mathrm{Re}(e^{i\omega t})\mathrm{Re}(\underline{S}^T)$$

$$- \mathrm{Im}(e^{i\omega t})\mathrm{Im}(\underline{S}^T)]\,dt \quad . \tag{3.25}$$

It is easily verified that

$$\int_0^T \mathrm{Re}(e^{i\omega t})\mathrm{Im}(e^{i\omega t})\,dt = 0 \quad . \tag{3.26}$$

Therefore, equation (3.25) can be rewritten as

$$\int_0^T \hat{\underline{x}}\,\hat{\underline{x}}^T\,dt = \mathrm{Re}(\underline{S})\mathrm{Re}(\underline{S}^T)\int_0^T \mathrm{Re}^2(e^{i\omega t})\,dt - \mathrm{Im}(\underline{S})\mathrm{Im}(\underline{S})\int_0^T \mathrm{Im}^2(e^{i\omega t})\,dt \qquad (3.26a)$$

It may readily be shown that

$$\int_0^T \mathrm{Re}^2(e^{i\omega t})\,dt = \int_0^T \cos^2(\omega t)\,dt = \int_0^T \mathrm{Im}^2(e^{i\omega t})\,dt$$

$$= \int_0^T \sin^2(\omega t)\,dt = \frac{T}{2}\ , \qquad (3.27)$$

Hence, equation (3.26a) yields

$$\int_0^T \hat{\underline{x}}\,\hat{\underline{x}}^T\,dt = \frac{T}{2}\left[\mathrm{Re}(\underline{S})\mathrm{Re}(\underline{S}^T) - \mathrm{Im}(\underline{S})\mathrm{Im}(\underline{S}^T)\right]\ . \qquad (3.28)$$

The left hand side of equation (3.18) can be manipulated to give

$$\int_0^T f_i\hat{\underline{x}}^T\,dt = \int_0^T f_i\mathrm{Re}(e^{i\omega t}\underline{S}^T)\,dt$$

$$= \int_0^T f_i\left[\mathrm{Re}(e^{i\omega t})\mathrm{Re}(\underline{S}^T) - \mathrm{Im}(e^{i\omega t})\mathrm{Im}(\underline{S}^T)\right]dt$$

$$= \left[\left(\int_0^T f_i\cos\omega t\,dt\right)\mathrm{Re}(\underline{S}^T)\right] - \left[\left(\int_0^T f_i\sin\omega t\,dt\right)\mathrm{Im}(\underline{S}^T)\right] \qquad (3.29)$$

Because of equations (3.28) and (3.24), equation (3.18) yields

$$\mathrm{Re}(\underline{S}^T)\int_0^T f_i \cos\omega t\,dt - \mathrm{Im}(\underline{S}^T)\int_0^T f_i \sin\omega t\,dt =$$

$$= [k_{i*}^e, c_{i*}^e]\frac{T}{2}[\mathrm{Re}(\underline{S})\mathrm{Re}(\underline{S}^T) - \mathrm{Im}(\underline{S})\mathrm{Im}(\underline{S}^T)]\ ;\ i = 1,\ldots,n\ . \qquad (3.30)$$

It is obvious that equation (3.30) is satisfied if

$$\mathrm{Re}(\underline{S}^T)\int_0^T f_i \cos\omega t\,dt = \frac{T}{2}[k_{i*}^e, c_{i*}^e]\mathrm{Re}(\underline{S})\mathrm{Re}(\underline{S}^T) \qquad (3.31)$$

and

$$\mathrm{Im}(\underline{S}^T)\int_0^T f_i \sin\omega t\,dt = \frac{T}{2}[k_{i*}^e, c_{i*}^e]\mathrm{Im}(\underline{S})\mathrm{Im}(\underline{S}^T)\ . \qquad (3.32)$$

Cancelling the terms $\mathrm{Re}(\underline{S}^T)$ and $\mathrm{Im}(\underline{S}^T)$ from equations (3.31) and (3.32) respectively yields

$$\frac{1}{T}\int_0^T f_i \cos\omega t\,dt = \tfrac{1}{2}[k_{i*}^e, c_{i*}^e]\mathrm{Re}(\underline{S})\ ;\quad i = 1,\ldots,n \qquad (3.33)$$

and

$$\frac{1}{T}\int_0^T f_i \sin\omega t\,dt = \tfrac{1}{2}[k_{i*}^e, c_{i*}^e]\mathrm{Im}(\underline{S})\ ;\quad i = 1,\ldots,n\ . \qquad (3.34)$$

Observing equations (3.33) and (3.34) it is realized that for every $i = 1,\ldots,n$ there exist at least two row vectors k_{i*} and c_{i*} for which equations (3.33) and (3.34) are satisfied. According to the preceding analysis, the existence of a solution of the above

equations is a sufficient condition to insure that equations (3.18) determining an equivalent linear system have a solution. Thus, an equivalent linear system always exists for the non-linear system (3.3). However, for $n \geq 2$ this system will be non-unique.

3.2.4 Application for Special Classes of Non-Linear Functions

3.2.4.1 General Remarks

The fact that an equivalent linear system exists, but may not be unique, allows a certain freedom in the selection of the system which ultimately will be used for obtaining an approximate steady-state periodic solution of the non-linear system (3.3). It is logical that among all the possible equivalent linear systems the most readily derivable should be selected for any specific non-linear problem. This possibility will be considered next in the context of a specific class of non-linear functions $\underline{f}(\underline{x}, \dot{\underline{x}})$.

Assume that the components of the vector $\underline{f}(\underline{x}, \dot{\underline{x}})$ satisfy the following conditions

$$f_i(\underline{x}, \dot{\underline{x}}) = \sum_{j=1}^{n} h_{ij}(y_{ij}, \dot{y}_{ij}) \tag{3.35}$$

$$f_i(0,0) = h_{ij}(0,0) = 0 \tag{3.36}$$

and

$$\int_0^T h_{ij} dt = 0 \quad , \tag{3.37}$$

where

$$y_{ij} = x_i - x_j \quad . \tag{3.38}$$

For conceptual purposes, equation (3.35) could be thought of as representing the non-linearity effect of a system of mass or nodal points interconnected by non-linear elements h_{ij} whose behavior depends only upon the relative coordinates between these points. Non-linear functions of the type presented above are often encountered in engineering practice. For instance, the non-linear function described by equation (3.35) may apply to the case of multi-story buildings representing the non-linear stiffness and damping of the structural elements. It is plausible to assume that these forces depend only on the relative displacement of the floors and its time derivative.

Before considering the determination of an equivalent linear system for systems with non-linear functions of the form (3.35), further discussion of property (3.37) is in order.

Averaging equation (3.3) over the interval $[0, T]$ yields

$$M \int_0^T \ddot{\underline{x}} dt + C \int_0^T \dot{\underline{x}} dt + K \int_0^T \underline{x} dt + \int_0^T \underline{f}(\underline{x}, \dot{\underline{x}})\, dt$$

$$= \underline{G} \int_0^T \mathrm{Re}(e^{i\omega t})\, dt \quad . \tag{3.39}$$

Upon using equation (3.5), equation (3.39) becomes

$$\int_0^T \underline{f}(\underline{x}, \dot{\underline{x}})\, dt = 0 \quad . \tag{3.40}$$

Thus, relation (3.40) must be satisfied if the solution $\underline{x}$ of the non-linear system (3.3) is of the form described by equation (3.4). It is easily verified that, if h_{ij} possesses property (3.37), then requirement (3.40) is satisfied. Hereafter, a non-linear function which possesses property (3.40) will be called symmetric. If a non-linear function does not possess property (3.40), it will be called non-symmetric. Figure [3.1.a] gives an example of a symmetric non-linear function, while Figure [3.1.b] shows a non-symmetric non-linear function. Systems with non-linearities of the latter type are occasionally encountered in ocean and automotive structures.

3.2.4.2 Systems with Symmetric Non-Linearities

Systems with symmetric non-linear functions (3.35) will be discussed herein. A systematic procedure for the determination of an equivalent linear system will be given.

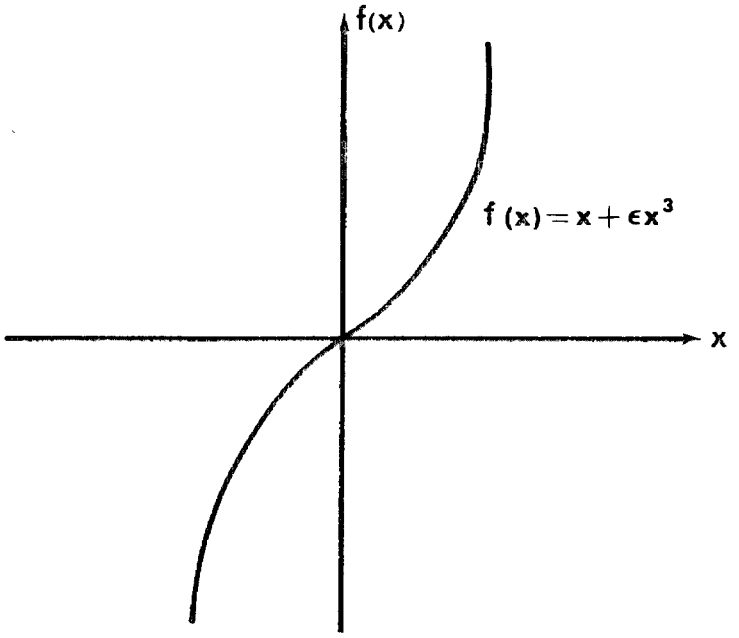
Because of equation (3.35), equations (3.33) and (3.34) can be respectively rewritten as

$$\sum_{j=0}^{n} \frac{1}{T} \int_0^T h_{ij} \cos \omega t \, dt = \tfrac{1}{2}[k_{i*}^e, c_{i*}^e] \mathrm{Re}(\underline{S}) \quad ; \quad i = 1, \ldots, n \qquad (3.41)$$

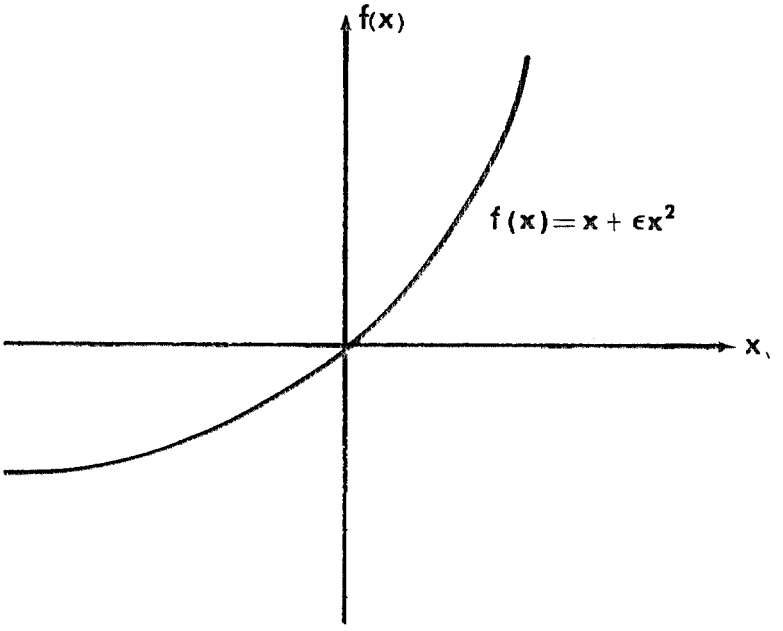
$$\sum_{j=0}^{n} \frac{1}{T} \int_0^T h_{ij} \sin \omega t \, dt = \tfrac{1}{2}[k_{i*}^e, c_{i*}^e] \mathrm{Im}(\underline{S}) \quad ; \quad i = 1, \ldots, n \; . \qquad (3.42)$$

Define the numbers γ_{ij} and $\varkappa_{ij}$; $i, k = 1, \ldots, n$, as solutions of the set of linear equations

$$\int_0^T h_{ij} \cos \omega t \, dt = \gamma_{ij} \int_0^T \dot{y}_{ij} \cos \omega t \, dt + \varkappa_{ij} \int_0^T y_{ij} \cos \omega t \, dt \qquad (3.43)$$



(a)



(b)

Figure 3.1. Symmetric Non-Linearity (a) and Non-Symmetric Non-Linearity (b)

and

$$\int_0^T h_{ij} \sin \omega t \, dt = \gamma_{ij} \int_0^T \dot{y}_{ij} \sin \omega t \, dt + \kappa_{ij} \int_0^T y_{ij} \sin \omega t \, dt \, . \quad (3.44)$$

Then, multiplying equations (3.43) by $\mathrm{Re}(X_i - X_j)$ and equation (3.44) by $\mathrm{Im}(X_i - X_j)$ and subtracting yields

$$\int_0^T h_{ij}[\cos \omega t \, \mathrm{Re}(X_i - X_j) - \sin \omega t \, \mathrm{Im}(X_i - X_j)] \, dt$$

$$= \gamma_{ij} \int_0^T \dot{y}_{ij}[\cos \omega t \, \mathrm{Re}(X_i - X_j) - \sin \omega t \, \mathrm{Im}(X_i - X_j)] dt$$

$$+ \kappa_{ij} \int_0^T y_{ij}[\cos \omega t \, \mathrm{Re}(X_i - X_j) - \sin \omega t \, \mathrm{Im}(X_i - X_j)] \, dt \, . \quad (3.45)$$

It is easily verified that

$$\cos \omega t \, \mathrm{Re}(X_i - X_j) - \sin \omega t \, \mathrm{Im}(X_i - X_j) = y_{ij} \, . \quad (3.46)$$

Therefore, equations (3.45) can be rewritten as

$$\int_0^T h_{ij} y_{ij} dt = \gamma_{ij} \int_0^T y_{ij} \dot{y}_{ij} dt + \kappa_{ij} \int_0^T y_{ij}^2 dt \, . \quad (3.47)$$

Evidently,

$$\int_0^T y_{ij} \dot{y}_{ij} dt = 0 \, . \quad (3.48)$$

Hence, equation (3.47) may be simplified as

$$\varkappa_{ij} = \frac{\int_0^T h_{ij} y_{ij} dt}{\int_0^T y_{ij}^2 dt} \quad ; \qquad i = 1, \ldots, n \ . \tag{3.49}$$

By an analogous procedure it is found that

$$\gamma_{ij} = \frac{\int_0^T h_{ij} \dot{y}_{ij} dt}{\int_0^T \dot{y}_{ij}^2 dt} \quad ; \qquad i = 1, \ldots, n \ . \tag{3.50}$$

It is recognized that equations (3.41) and (3.42) simply represent decompositions of the quantities $\int_0^T f_i \cos \omega t \, dt$ and $\int_0^T f_i \sin \omega t \, dt$ in linear combinations of the amplitudes $|X_i|$; $i = 1, \ldots, n$. Hence, the set of equivalent linear elements given by equations (3.49) and (3.50) constitutes an acceptable solution of equations (3.41) and (3.42). It is understood that in this case the elements c_{ij}^e and k_{ij}^e of the matrices C_e and K_e will be linear combinations of γ_{ij} and $\varkappa_{ij}$; $i, j = 1, \ldots, n$.

An important fact must be noted herein. According to the classical method of equivalent linearization, the elements γ_{ij} and $\varkappa_{ij}$ given by (3.49) and (3.50) are the equivalent linear damping and stiffness for the non-linear function $h_{ij}(y_{ij}, \dot{y}_{ij})$. This means that

for the equivalent linear system defined by equations (3.49) and (3.50), the minimization criterion (3.9) is both satisfied for the whole system (3.3) and for each interconnecting non-linear element h_{ij}. In order to summarize the results of the preceding analysis the following result is stated.

Main Result

Given the dynamical system (3.3) with symmetric non-linear functions of the form (3.35), an equivalent linear system (3.6) may be obtained by linearizing each interconnecting element $h_{ij}(y_{ij}, \dot{y}_{ij})$ according to the rule

$$h_{ij}(y_{ij}, \dot{y}_{ij}) \rightarrow \gamma_{ij}\dot{y}_{ij} + \kappa_{ij}y_{ij}$$

where

$$\gamma_{ij} = \frac{\int_0^T h_{ij}\dot{y}_{ij}dt}{\int_0^T \dot{y}_{ij}^2 dt} \quad \text{and} \quad \kappa_{ij} = \frac{\int_0^T h_{ij}y_{ij}dt}{\int_0^T y_{ij}^2 dt} . \qquad (3.51)$$

3.2.4.3 Systems with Non-Symmetric Non-Linearities

At this point attention is focused on dynamical systems with non-symmetric non-linearities.

A single-degree-of-freedom oscillator with a non-symmetric non-linear function $f(x) = bx^2$, $b > 0$ has been discussed in reference [43]. There, it is found through perturbation techniques that the

steady-state periodic solution of such a system exhibits a constant offset, the value of which depends on the amplitude of the oscillatory component of the response. Following this general result, it is postulated that the steady-state periodic solution of system (3.3) with non-symmetric $\underline{f}(\underline{x},\dot{\underline{x}})$ will be of the form

$$\underline{x}(t) = (x_1,\ldots,x_n)^T \quad , \tag{3.52}$$

where

$$x_i(t) = X_{ci} + \mathrm{Re}(X_i e^{i\omega t}) \quad ; \qquad i = 1,\ldots,n \quad . \tag{3.53}$$

The numbers X_{ci} and X_i are real and complex respectively.

Averaging equation (3.3) over the interval $[0,T]$ gives

$$K\underline{X}_c + \frac{1}{T}\int_0^T \underline{f}(\underline{x},\dot{\underline{x}}) = 0 \tag{3.54}$$

where

$$X_c^T = (X_{c1},\ldots,X_{cn}) \quad . \tag{3.55}$$

Define the vector $\underline{z}(t)$ by

$$\underline{x}(t) = \underline{X}_c + \underline{z}(t) \quad . \tag{3.56}$$

Substituting transformation (3.56) into equation (3.3) yields

$$M\ddot{\underline{z}} + C\dot{\underline{z}} + K\dot{\underline{z}} + K\underline{X}_c + \underline{f}(\underline{x}_c + \underline{z}, \dot{\underline{z}}) = \mathrm{Re}(e^{i\omega t})\underline{G} \quad . \tag{3.57}$$

Define the function $\underline{\beta}$ as

$$\underline{\beta}(\underline{X}_c, \underline{z}, \dot{\underline{z}}) = K\underline{X}_c + \underline{f}(\underline{X}_c + \dot{\underline{z}}, \dot{\underline{z}}) \quad . \tag{3.58}$$

Then, because of condition (3.54) it is seen that $\underline{\beta}$ is symmetric with respect to $\underline{z}$. Hence, all of the preceding analysis pertaining to systems with symmetric non-linearities may be applied for system (3.57).

As an illustration of the method, an equivalent linear system for the non-linear system (3.3) with non-symmetric function $f(\underline{x}, \dot{\underline{x}})$ of the form (3.35) will be determined.

The quantities φ_{ij} are defined as

$$\varphi_{ij} = z_i - z_j \; ; \qquad i,j = 1,\ldots,n \quad . \tag{3.59}$$

It is understood that because of the extra term $K\underline{X}_c$ in (3.58), an equivalent linear system cannot be determined by direct application of formulae (3.49) and (3.50). Thus, recoursing to equations (3.41) is necessary. The integrals I_{ci} and I_{si} are defined as

$$I_{ci} = \frac{1}{T}\int_0^T \beta_i \cos \omega t \tag{3.60}$$

and $\qquad i = 1,\ldots,n$

$$I_{si} = \frac{1}{T}\int_0^T \beta_i \sin \omega t \quad , \tag{3.61}$$

where

$$\underline{\beta}^T = (\beta_1,\ldots,\beta_n) \quad . \tag{3.62}$$

Equations (3.41) may be written in terms of I_{ci} and I_{si} as

$$I_{ci} = \tfrac{1}{2}[k^e_{i*}, c^e_{i*}]\mathrm{Re}(\underline{S}) \quad ; \qquad i = 1, \ldots, n \tag{3.63}$$

and

$$I_{si} = \tfrac{1}{2}[k^e_{i*}, c^e_{i*}]\mathrm{Im}(\underline{S}) \quad ; \qquad i = 1, \ldots, n \ . \tag{3.64}$$

Using equation (3.58) in which the vector function $\underline{\beta}(\underline{X}_c, \underline{z}, \dot{\underline{z}})$ is defined, expressions (3.60) and (3.61) for the integrals I_{ci} and I_{si} may be simplified as follows

$$I_{ci} = \frac{1}{T}\int_0^T \left(\sum_{j=1}^{n} k_{ij}X_{cj} + \sum_{j=1}^{n} h_{ij}\right)\cos \omega t \, dt$$

$$= \sum_{j=1}^{n} \frac{1}{T}\int_0^T h_{ij} \cos \omega t \, dt \tag{3.65}$$

and

$$I_{si} = \frac{1}{T}\int_0^T \left(\sum_{j=1}^{n} k_{ij}X_{cj} + \sum_{j=1}^{n} h_{ij}\right)\sin \omega t \, dt$$

$$= \sum_{j=1}^{n} \frac{1}{T}\int_0^T h_{ij} \sin \omega t \, dt \ . \tag{3.66}$$

Because of the final forms of the integrals I_{ci} and I_{si}, it is clear that an equivalent linear system can be constructed for the system with non-symmetric non-linear function in the same manner as was done for the system with a symmetric non-linear function. Specifically, the equivalent linear elements γ_{ij} and κ_{ij} are defined by the set of equations

$$\int_0^T h_{ij} \cos \omega t \, dt = \gamma_{ij} \int_0^T \dot{\varphi}_{ij} \cos \omega t \, dt + \varkappa_{ij} \int_0^T \varphi_{ij} \cos \omega t \, dt \quad (3.67)$$

and

$$\int_0^T h_{ij} \sin \omega t \, dt = \gamma_{ij} \int_0^T \dot{\varphi}_{ij} \sin \omega t \, dt + \varkappa_{ij} \int_0^T \varphi_{ij} \sin \omega t \, dt \, . \quad (3.68)$$

Hence,

$$\varkappa_{ij} = \frac{\int_0^T h_{ij} \varphi_{ij} dt}{\int_0^T \varphi_{ij}^2 dt} \quad ; \qquad i, j = 1, \ldots, n \quad (3.69)$$

and

$$\gamma_{ij} = \frac{\int_0^T h_{ij} \dot{\varphi}_{ij} dt}{\int_0^T \dot{\varphi}_{ij}^2 dt} \, . \quad (3.70)$$

It is clear that the offset vector provides only n compensating parameters which must be used to satisfy condition (3.54). Obviously, the number of distinct non-linear elements h_{ij} is $n(n-1/2)$. Thus, it is clear that in general condition (3.54) cannot be satisfied for each interconnecting element h_{ij}, but only componentwise for the whole vector $\underline{f}(\underline{x}, \dot{\underline{x}})$, as in fact is done in equation (3.54). For the special case of chainlike systems where

$$h_{ij} = 0 \quad \text{if} \quad |i-j| \geq 2 \tag{3.71}$$

condition (3.54) can be satisfied by determining the offset vector $\underline{X}_c$ so that this condition is satisfied for every interconnecting element h_{ij}.

The preceding analysis of the case of non-symmetric non-linear functions may be summarized in the following result.

Main Result

Given the dynamical system (3.3) with a non-symmetric non-linear function of the form (3.35), an equivalent linear system may be obtained by replacing each interconnecting element according to the rule

$$h_{ij}(y_{ij}, \dot{y}_{ij}) \rightarrow \gamma_{ij}\dot{y}_{ij} + \kappa_{ij}y_{ij} \tag{3.72}$$

where

$$\gamma_{ij} = \frac{\int_0^T h_{ij}(y_{ij}, \dot{y}_{ij})\dot{\varphi}_{ij}dt}{\int_0^T \dot{\varphi}_{ij}^2 dt} \quad \text{and} \quad \kappa_{ij} = \frac{\int_0^T h_{ij}(y_{ij}, \dot{y}_{ij})\varphi_{ij}dt}{\int_0^T \varphi_{ij}^2 dt} .$$

An offset vector $\underline{X}_c$ must be included in its steady-state periodic solution $\underline{x}(t)$ (3.53) so that the condition (3.54) of zero average non-linear function is satisfied.

3.2.5 Mechanization of the Method

For non-linear systems (3.3) with symmetric non-linearities the mechanization of the method is well understood [44,45]. Herein a brief outline of the manner in which the method may be applied for non-linear systems with non-symmetric non-linearities will be given.

The lack of "symmetricity" of any of the components f_i; $i = 1,\ldots,n$, of the non-linear function $\underline{f}(\underline{x},\dot{\underline{x}})$ necessitates the addition of an offset vector $\underline{X}_c$ to the oscillatory part of the approximate steady-state periodic solution of the problem. Therefore, the identification parameters of the solution are the quantities

$$X_{ci}\ ,\ X_{oi} \text{ and } \theta_i\ ; \qquad i = 1,\ldots,n$$

where

$$\underline{x}(t) = [x_1(t),\ldots,x_n(t)]^T \tag{3.73}$$

and

$$x_i(t) = X_{ci} + X_{oi}\cos(\omega t + \theta_i)\ ; \qquad i = 1,\ldots,n\ . \tag{3.74}$$

The ultimate goal of the method is to determine the quantities X_{ci}, X_{oi} and θ_i; $i = 1,\ldots,n$. An initial relation between X_{ci} and X_{oi} is given by condition (3.54) which insures the zero average of non-linear function. By transformation (3.56) the original non-linear system (3.3) is written as (3.57). For this system an equivalent linear system can always be constructed by selecting any two matrices C_e and K_e which satisfy the indeterminate system of equations (3.33) and (3.34). If the non-linearity of the system is of the form (3.35), an equivalent linear system can be constructed

directly according to equation (3.72). As it was indicated in the general analysis of the method performed in Chapter II, the parameters of the equivalent linear system depend in general on the quantities X_{oi}, X_{ci} and θ_i; $i = 1, \ldots, n$.

Next the steady-state periodic solution of the equivalent linear system (3.6) must be generated. This is readily accomplished by substituting in the equivalent linear system the expression for the solution vector $\underline{x}(t)$ and equating the coefficients of the functions $\cos \omega t$ and $\sin \omega t$ to zero. By this procedure 2n algebraic equations in terms of X_{oi}, X_{ci} and θ_i; $i = 1, \ldots, n$, are obtained. The above equations coupled with the conditions of zero average non-linear function constitute a set of 3n algebraic equations with 3n unknowns. It is rare that these equations can be solved analytically. It is most probable that recoursing to a digital computer will be necessary. For this purpose, methods of modern numerical analysis, as Newton's techniques and its modifications, are readily available. It is indisputable that the computer time which will be consumed for the numerical calculation of the amplitudes, the phases, and the offset parameters will be overwhelmingly less than the computer time which is necessary for a direct integration of the non-linear system (3.3). This is due to the fact that the determination of the amplitudes, the phases, and the offset parameters of the response by way of generalized equivalent linearization, requires the solution of 3n non-linear algebraic equations, while the direct integration of system (3.3) must be carried out up to an appropriately long time so that the solution becomes periodic.

3.2.6 Accuracy of the Method

In this disseration no work towards the examination of the accuracy of the method was done. For the case of a single-degree-of-freedom non-linear oscillator, interesting analytical work toward this end has been done in references [46] and [47]. In general, application of either contraction mapping or iterative procedures leads to a bound on the solution error. In reference [46] it was found that the above techniques applied to the case of a Duffing oscillator significantly overestimated the actual solution error determined by direct numerical integration.

In reference [45] the accuracy of the method was examined for the case of a uniform ten-mass chainlike structure with a bilinear softening terminal. The approximate response was compared with the "exact" solution found by direct numerical computation. According to this analysis the difference between the "exact" and the approximate solution was less than 5% in both amplitude and frequency.

It is worthy to comment on the relation between the method of equivalent linearization and the perturbation techniques. Usually, in problems which are amenable to perturbation analysis the non-linear function is multiplied by a small parameter. For a single-degree-of-freedom non-linear oscillator subjected to harmonic excitation, the solutions obtained by the method of equivalent linearization and by a first-order perturbation analysis are identical [14]. Therefore, if the herein discussed small parameter is sufficiently small, the method of equivalent linearization furnishes a very accurate solution.

However, this is only a sufficient condition and should not be considered as a limitation of the method, since it has been formulated independently of any concept of perturbation.

3.2.7 Summary

The applicability of the generalized method of equivalent linearization to multi-degree-of-freedom non-linear systems has been examined. The system is subjected to harmonic mono-frequency excitation and an approximate steady-state harmonic response is sought. The general formulation of the method, presented in Chapter II, is applied for the present case using as operator G the average over one period of the solution. For systems with symmetric non-linearities the identification parameters of the solution vector are the amplitudes and the phases of its components. For systems with non-symmetric non-linearities a constant offset vector is added to the oscillatory part of the response.

It is found that a non-unique equivalent linear system always exists. All the equivalent linear systems will result in the same value of the quantity $\frac{1}{T}\int_0^T \underline{d}^T \underline{d}\, dt$. For spring-mass systems with symmetric non-linearities an equivalent linear system may be constructed by linearizing independently each interconnecting non-linear element. Modification of the above rule to account for non-symmetric non-linear functions is made as well. The mechanization of the method for the latter type of systems is briefly outlined.

3.3 Stationary Random Response

3.3.1 Non-Linear System

Herein the stationary random response of a multi-degree-of-freedom non-linear system which is excited by a stationary Gaussian random vector $\underline{G}(t)$ is considered. For this special case the general equation (2.1) can be written as

$$M\ddot{\underline{x}} + C\dot{\underline{x}} + K\underline{x} + \underline{f}(\underline{x},\dot{\underline{x}}) = \underline{G} \ , \tag{3.75}$$

where

$$\underline{G}^T = (g_1,\ldots,g_n) \ . \tag{3.76}$$

The components g_i; $i = 1,\ldots,n$, of the vector $\underline{G}$ are stationary Gaussian processes.

An exact method for studying the stationary random response of non-linear dynamical systems is the solution of the associated Fokker-Planck equation. If the excitation is a Gaussian white noise, the transition probability density of the response satisfies the Fokker-Planck equation. It is well known that this transition probability density completely defines the response process. Unfortunately at present there is no systematic way of solving the complete Fokker-Planck equation for every second order non-linear system.

Because of the scarcity of exact solutions of problems of practical importance, attention has been turned to methods of approximate analysis. If the effect of the non-linearity on the overall response of the system is small, several approximate techniques can be used. One of these is the normal mode approach

in which an approximate solution is generated by solving several single-degree-of-freedom non-linear systems. This method is not generally applicable and certain rather restrictive conditions must be satisfied by the system and the excitation as well [20].

Another technique for generating an approximate solution is the perturbation approach [22]. This method is suitable for problems in which the non-linear terms of the system are small compared to the linear terms, and the level of excitation is also sufficiently small. The method is a direct adaptation of the well established technique for non-linear systems under deterministic excitation. According to this method the exact solution is expressed as a power series in the non-linear parameter. The coefficients of the series are determined by solving linear differential equations. These equations are derived by substituting the power series representation of the solution into the original equation and equating the coefficients of line powers of the nonlinearity parameter. To date, there exists no proof of convergence of the above mentioned series. Furthermore, the non-linear parameter must be quite small if reliable results are to be obtained.

In the next section the manner in which the method of generalized equivalent linearization may be applied to system (3.75) will be outlined.

3.3.2 Equivalent Linearization

It is assumed that the non-linear system (3.75) possesses a stationary solution.

The equivalent linear system is defined by the linear differential equation

$$M\ddot{\underline{x}} + (C + C_e)\dot{\underline{x}} + (K + K_e)\underline{x} = \underline{G} \ , \qquad (3.77)$$

where the matrices C_e and K_e must be such that the difference $\underline{d}$ between systems (3.76) and (3.77) is minimized for every $\underline{x}$ which belongs to the class of solutions of system (3.78). The difference $\underline{d}$ is defined by

$$\underline{d} = \underline{f}(\underline{x}, \dot{\underline{x}}) - C_e\dot{\underline{x}} - K_e\underline{x} \ . \qquad (3.78)$$

Since the excitation $\underline{G}$ of the linear system is assumed Gaussian, it is well known that the response $\underline{x}$ will be Gaussian as well. Therefore, the matrices C_e and K_e must be such that the difference $\underline{d}$ is minimized for every stationary Gaussian random vector $\underline{x}$.

Clearly, all the steps of the procedure presented in the development of the general method apply herein. Consequently, only "technical" details remain to be discussed.

The Euclidean norm $(\underline{d}^T\underline{d})^{\frac{1}{2}}$ of $\underline{d}$ is used in the procedure of minimization of $\underline{d}$. As averaging operator $\mathbb{G}$, the operator E which yields the ensemble mean is used. Obviously, this operator possesses all the properties which were discussed in (2.3.6). The final form of the minimization criterion is

$$E(\underline{d}^T\underline{d}) = \text{minimum} \ . \qquad (3.79)$$

According to section (2.3.4) the necessary conditions for (3.79) to be true are

$$\frac{\partial}{\partial c_{ij}^{e}} (D_i^2) = 0 \; ; \qquad j = 1, \ldots, n \tag{3.80}$$

$$\frac{\partial}{\partial k_{ij}^{e}} (D_i^2) = 0 \; ; \qquad j = 1, \ldots, n \tag{3.81}$$

where

$$D_i^2 = E(d_i^2) \; ; \qquad i = 1, \ldots, n \tag{3.82}$$

and

$$\underline{d}^T = (d_1, \ldots, d_n) \; . \tag{3.83}$$

By virtue of the analysis given in (2.3.4), equations (3.80) and (3.81) can be rewritten as

$$E[f_i \underline{\hat{x}}] = E[\underline{\hat{x}}\, \underline{\hat{x}}^T] \begin{bmatrix} k_{i*}^{e\,T} \\ c_{i*}^{e\,T} \end{bmatrix} \; ; \qquad i = 1, \ldots, n \tag{3.84}$$

where

$$\underline{\hat{x}} = \begin{pmatrix} \underline{x} \\ \underline{\dot{x}} \end{pmatrix} \tag{3.85}$$

and k_{i*}^{e}, c_{i*}^{e} are the $\underline{i}\underline{th}$ rows of the matrices K_e and C_e respectively.

The question of whether or not a solution of the set of equations (3.84) makes the quantity $E(\underline{d}\, \underline{d}^T)$ a true minimum has been

answered in (2.3.5) for the case of a general averaging operator G. According to the general result of (2.3.5) the value of $E(\underline{d}^T\underline{d})$ corresponding to any matrices C_e and K_e which satisfy equation (3.84) is not bigger than the value of $E(\underline{d}^T\underline{d})$ corresponding to any other pair of matrices C_e and K_e. Therefore, any linear system (3.77) with matrices C_e and K_e satisfying equation (3.84) is consistent with the philosophy of the equivalent linearization method.

The existence and uniqueness of the equivalent linear system depends on the nature of the matrix $E(\hat{\underline{x}}\,\hat{\underline{x}}^T)$. If the matrix $E(\hat{\underline{x}}\,\hat{\underline{x}}^T)$ is non-singular, there exists a unique equivalent linear system. If this matrix is singular, an equivalent linear system, if one exists, is non-unique.

According to the general result of (2.3.6), adapted for the operator E and the class of Gaussian random vectors $\hat{\underline{x}}$, the matrix $E(\hat{\underline{x}}\,\hat{\underline{x}}^T)$ is singular if and only if the components of the vector $\hat{x}$ are linearly dependent. This possibility cannot be excluded a priori. Consequently, the matrix $E(\hat{\underline{x}}\,\hat{\underline{x}}^T)$ may be singular for some Gaussian random vector $\hat{\underline{x}}$. This fact implies obvious complexities if a numerical approach is used for the determination of the matrices C_e and K_e.

The direct numerical solution of equation (3.84) was first proposed in reference [23]. Later, analytic formulae for the determination of C_e and K_e were given in reference [9]. The approach presented in the last reference is well suited to spring-mass systems with non-linear functions $\underline{f}(\underline{x},\dot{\underline{x}})$ of the form (3.35). For this type of system it was proved that equation (3.84) is satisfied by the matrices C_e

and K_e which are obtained by applying the method of equivalent linearization to each interconnecting element h_{ij}. Each of these non-linear elements depends solely on the relative displacement y and velocity $\dot{y}_{ij}$ of the masses of the system.

Specifically, using the mathematical properties of the multi-dimensional Gaussian distribution, it was proved in reference [9] that each non-linear element $h_{ij}(y_{ij}, \dot{y}_{ij})$ may be replaced by a combination of linear elements according to the rule

$$h_{ij}(y_{ij}, \dot{y}_{ij}) \rightarrow \gamma_{ij}\dot{y}_{ij} + \kappa_{ij}y_{ij} \tag{3.86}$$

where

$$\gamma_{ij} = E[h_{ij}(y_{ij}, \dot{y}_{ij})\dot{y}_{ij}]/E(\dot{y}_{ij}^2) \tag{3.87}$$

and

$$\kappa_{ij} = E[h_{ij}(y_{ij}, \dot{y}_{ij})y_{ij}]/E(y_{ij}^2) \quad . \tag{3.88}$$

Recently, a direct analytic formula for the determination of the matrices C_e and K_e was given in reference [24]. This formula is applicable for every non-linear system with single-valued non-linear function $\underline{f}(\underline{x}, \dot{\underline{x}})$ subjected to a Gaussian random vector. Specifically it was proved that, if a pair of matrices C_e and K_e is constructed with elements c_{ij} and k_{ij} given by

$$c_{ij} = E\left[\frac{\partial f_i(\underline{x}, \dot{\underline{x}})}{\partial \dot{x}_j}\right] \quad ; \qquad i, j = 1, \ldots, n \tag{3.89}$$

and

$$k_{ij} = E\left[\frac{\partial f_i(\underline{x}, \dot{\underline{x}})}{\partial x_j}\right] \quad ; \qquad i,j = 1,\ldots,n \tag{3.90}$$

then equation (3.84) is satisfied and an equivalent linear system is found.

The approaches of both references [9] and [24] are mainly based on the special form of the probability density of the multi-dimensional Gaussian distribution,

$$p(\hat{\underline{x}}) = (2\pi)^{2n} \det(\Omega^{-1}) \exp(-\tfrac{1}{2}\hat{\underline{x}}^T \Omega \hat{\underline{x}}) \ , \tag{3.91}$$

where

$$\Omega = E(\hat{\underline{x}}\,\hat{\underline{x}}^T) \ . \tag{3.92}$$

Prolonged efforts of the writer to derive either equations (3.87) and (3.88) or equations (3.89) and (3.90) by the same technique which was used for the deterministic case were not fruitful.

From equations (3.91) and (3.92) it is obvious that the parameters of the equivalent linear system (3.77) depend on the statistics of the response. If during the determination of the elements of the equivalent linear system higher order joint moments appear, they can be expressed in terms of the second order moments by repeated application of the relation

$$E[\underline{y}\, q(\underline{y})] = E(\underline{y}\,\underline{y}^T)\, E[\nabla q(\underline{y})] \tag{3.93}$$

where

$$\nabla^T = \left[\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right] \tag{3.94}$$

and $\underline{y}$ is any jointly Gaussian random vector with zero mean. This relation is proved in reference [24].

Upon determination of the matrices C_e and K_e, the linear system (3.77) can be solved by any of a number of available analytical techniques. This procedure will yield non-linear algebraic equations in terms of the second order moments of the response and its time derivatives. At this stage the procedure of the equivalent linearization has been completed. It is noted that the matrix $E(\hat{\underline{x}}\,\hat{\underline{x}}^T)$ totally describes all the statistics of the stationary Gaussian solution vector $\hat{\underline{x}}$. Depending on the complexity of the algebraic equations, closed form or numerical solutions may be obtained.

Generally, approximate solutions obtained by the generalized equivalent linearization technique are most dependable for weakly non-linear systems. However, it has been reported [9] that the method produces reasonable results even for some severe non-linearities. Specifically, in reference [9] the method was applied to spring-mass multi-degree-of-freedom chain-like non-linear system with a cubic hardening non-linearity. In that investigation, the difference between the exact solution and the approximate solution was reported to be no more than 7.5% even for very large values of the coefficient of the cubic non-linearity.

IV. Application to Transient Response of Single-Degree-of-Freedom Non-Linear Systems

4.1 Introduction

In Chapter III the method of equivalent linearization was used to provide algebraic equations for the identification parameters of the steady-state or stationary response of a multi-degree-of-freedom non-linear system. For the case of monofrequency harmonic excitation, the identification parameters were the amplitudes and the phases of the vector response. For the case of stationary Gaussian vector excitation, the identification parameters were the elements of the covariance matrix of the response.

In this chapter the method of equivalent linearization will be utilized for the derivation of differential equations which describe the change of certain parameters of the response in terms of the independent variable of the specific problem. Certain assumptions will be made for the dynamical systems under consideration. These assumptions are plausible for a wide range of practical engineering problems. The potentiality of such an approach is evident. Approximate transient solutions of certain parameters of the response can be derived. Through this analysis various important questions of practical interest can be answered. For example, the time required for the system to reach its steady-state response may be determined.

In the following sections the equivalent linearization will be used specifically for the analysis of the transient response of a lightly damped single-degree-of-freedom non-linear oscillator. Both

the cases of deterministic harmonic and stationary Gaussian white excitation will be considered.

Since the system under consideration is assumed to be lightly damped and weakly non-linear, it can be proved that its response exhibits pseudo-sinusoidal behavior [12,48]. The amplitude and the phase of the response are not constant but slowly varying functions of the independent variable of the problem. For apparent reasons, knowledge of the amplitude of the response is highly desirable for the engineering applications. Equations which describe the amplitude of the transient response will be derived.

4.2 Harmonic Excitation

4.2.1 General Remarks

Herein the transient response of a lightly damped and weakly non-linear dynamical system subjected to harmonic excitation will be considered.

Usually, the "weakness" of the non-linearities is represented by a small parameter which is the coefficient of the non-linear function of the system. For this classical problem several versions of the perturbation techniques originally introduced by Poincaré are available. If a first-order steady-state solution only is desired, the problem may be analyzed by the method of equivalent linearization [14]. Herein, it will be shown that the method of equivalent linearization can be used for the analysis of the transient solution of the problem as well. The results will be compared with those generated by the perturbation method.

4.2.2 Formulation

For the purpose of the present discussion it will be assumed that the non-linear equation of motion has the form

$$m\ddot{x} + c\dot{x} + kx + \epsilon f(x, \dot{x}) = F \cos vt \tag{4.1}$$

where $\epsilon |f(x, \dot{x})|$ is small compared to $c\dot{x} + kx$ for all x and $\dot{x}$. The last condition may be interpreted as requiring that the solution of the non-linear problem differs only slightly from that of the linear problem obtained by neglecting all non-linear terms. The following additional assumptions about the damping of the system c, the angular frequency v and the amplitude F of the excitation are made

$$\frac{c}{m} = 0(\epsilon) \tag{4.2}$$

$$v^2 = \frac{k}{m} [1 + 0(\epsilon)] \tag{4.3}$$

where

$$\epsilon \ll 1 \tag{4.4}$$

and

$$\frac{F}{m} = 0(\epsilon) \ . \tag{4.5}$$

In reference [49] it is shown that the external force contributes to the first approximation of the solution of equation (4.1) only if condition (4.3) is satisfied.

In view of the smallness of the external force, the damping coefficient and the non-linear term, the oscillation during a single cycle will be nearly harmonic. That is

$$x = a \cos (vt + \theta) \tag{4.6}$$

and

$$\frac{dx}{dt} = \dot{x} = -av \sin (vt + \theta) \tag{4.7}$$

where a and θ are slowly varying functions of the variable t.

If equations (4.6) and (4.7) are solved, considering as unknown quantities the variables a and θ, the following relations are obtained

$$a^2 = x^2 + \frac{\dot{x}^2}{v^2} \tag{4.8}$$

and

$$\theta = \tan^{-1}\left(\frac{\dot{x}}{vx}\right) - vt \ . \tag{4.9}$$

It is recognized that the right hand side of (4.8) is proportional to the total energy of the system. The mechanism of the viscous damping, the non-linearity and the external excitation are the only causes of any change of the energy of the system. Since the coefficient of the damping, the parameter ϵ, and the amplitude of the exciting force were assumed to be small, the rate of change of the energy of the system will be small. Therefore, the rate of change of the amplitude a is small which means that a is a slowly varying function of the independent variable t.

Analogous analysis can be made for the phase angle θ. From equation (4.9) it is clear that θ represents the phase difference between the excitation and the solution of system (4.1). Actually, since v is assumed to be close to the natural angular frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{4.10}$$

of the system, θ represents the "detuning" of the solution from ω_0. The only terms of (4.1) which could cause any change of θ are the damping mechanism, the non-linearity, and the excitation. Hence, by the same argument given for the amplitude a, the phase θ is a slowly varying function of t. In fact, in reference [12] it is proved that

$$\dot{a} = 0(\epsilon) \tag{4.11}$$

and

$$\dot{\theta} = 0(\epsilon) \quad . \tag{4.12}$$

According to the preceding analysis, the amplitude a and the phase θ of the approximate solution can be assumed to be almost constant during one cycle of oscillation. The solution x and its derivative $\dot{x}$ are given by (4.6) and (4.7) respectively. Since the form of the approximate solution is known, the construction of an equivalent linear system for the non-linear system (4.1) is a straightforward matter. The non-linear function $\epsilon f(x, \dot{x})$ can be replaced according to the rule [14]

$$\epsilon f(x, \dot{x}) \rightarrow \epsilon c_e \dot{x} + \epsilon k_e x \quad . \tag{4.13}$$

where

$$c_e = -\frac{1}{v\pi a} \int_0^{2\pi} f(a \cos \psi, -av \sin \psi) \sin \psi \, d\psi \tag{4.14}$$

and

$$k_e = \frac{1}{\pi a} \int_0^{2\pi} f(a \cos \psi, -av \sin \psi) \cos \psi \, d\psi \quad . \tag{4.15}$$

Hence, the equivalent linear system of (4.1) is

$$m\ddot{x} + (c + \varepsilon c_e)\dot{x} + (k + \varepsilon k_e)x = F \sin vt \tag{4.16}$$

with solution x and its derivative $\dot{x}$ given by (4.6) and (4.7).

Next, the equivalent linear system (4.16) will be used for the derivation of differential equation which governs the amplitude of the transient response of system (4.1). For this purpose the energy of the equivalent linear system is considered.

During one cycle of oscillation, energy is imparted to the system by the external excitation and energy is dissipated out through the damping mechanism. The difference between these two energies represents the change of the total energy of the system. The energy per unit mass which is imparted to the system by the excitation may be expressed as

$$E_{excitation} = \int_0^{\frac{2\pi}{v}} \frac{F}{m} \sin vt \dot{x} dt = -\pi a \frac{F}{m} \cos \theta \quad . \tag{4.17}$$

The energy per unit mass which is dissipated from the system through the damping mechanism may be expressed as

$$E_{dissipated} = \int_0^{\frac{2\pi}{v}} \frac{c + \varepsilon c_e}{m} \dot{x}^2 dt = \pi v \frac{a^2}{m} (c + \varepsilon c_e) \quad . \tag{4.18}$$

At the end of one full cycle of oscillation the total energy per unit mass of the system will have changed by an amount equal

$$\Delta E_{total} = \frac{v^2}{2}(a + \Delta a)^2 - \frac{v^2}{2} a^2 \approx a\Delta a v^2 \quad . \tag{4.18}$$

Clearly, the energy balance equation of the system can be written as

$$\Delta E_{total} = E_{excitation} - E_{dissipation} \quad . \tag{4.19}$$

Upon using the corresponding expressions for the energies, equation (4.19) can be written as

$$v^2 a\Delta a = -\pi a \frac{F}{m} \cos\theta - \pi v \frac{a^2}{m}(c + \varepsilon c_e) \quad . \tag{4.20}$$

The change Δa of the amplitude a during one cycle of oscillation with period $T = 2\pi/v$ is small. Consequently, the derivatives of a with respect to the independent variable t can be approximated by

$$\frac{da}{dt} \approx \frac{\Delta a}{T} = \frac{\Delta a}{2\pi/v} \quad . \tag{4.21}$$

Because of equation (4.21), equation (4.20) can be written as

$$\frac{da}{dt} = -\frac{c + \varepsilon c_e(a)}{2m} a - \frac{F}{2mv} \cos\theta \quad . \tag{4.22}$$

Equation (4.22) describes the amplitude response of system (4.1).

Equation (4.22) agrees to the first order of ε with the equation derived in reference [12] for the amplitude a of the response of system (4.1). Reference [12] approaches the problem by classical perturbation techniques.

During the generation of this dissertation an equation which governs the change of the phase θ was also derived. This equation agrees to the first order in ε with the equation derived in reference [12] by perturbation techniques. However, the procedure of the derivation of the equation governing $\dot{\theta}$ by considering the equivalent linear system was not as direct as that given above for $\dot{a}$. Specifically, it was necessary to use the mathematical relationship between $a, \dot{a}, \theta$ and $\dot{\theta}$ which is implied by assumptions (4.6) to relate $a, \dot{a}, \theta$ and $\dot{\theta}$. Subsequently, the mechanisms of energy input and dissipation associated with the equivalent linear system were considered. Finally, the principle of averaging was used to derive an equation for $\dot{\theta}$. Interpretation of the derived equation in the context of the reactive energy of the equivalent linear system was not successful. It is felt that the analysis for the phase θ does not add appreciably to the understanding of the herein presented method. Therefore, it is not included in this thesis.

4.2.3 Summary

The purpose of the preceding analysis was to further indicate the potential of the method of equivalent linearization. Previously, the method was applied only for approximation of the steady-state response of non-linear dynamical systems. This approach yields

non-linear algebraic equations for the amplitudes and the phases of the steady-state approximate response. The present extension of the method allows application of the method for the derivation of equations which govern the transient response of the system.

The lightly damped and weakly non-linear system (4.1) has been considered herein. The response of this type of system exhibits pseudo-sinusoidal behavior with amplitude a and phase θ. For this class of solution functions the equivalent linear system (4.14) is constructed. The equivalent linear elements are given by equations (4.12) and (4.13). By consideration of the energy balance of system (4.14) over one cycle of oscillation, differential equation (4.22) for the amplitude is derived. This equation agrees to first order with the equation which is derived by classical perturbation techniques. Therefore, it is possible to avoid many of the calculational complexities of the perturbation techniques by using the method of equivalent linearization and applying energy principles. In addition to overcoming the calculational burden of the perturbation techniques,it is thought that the herein presented approach provides a better "engineering" understanding of what is really happening in the system.

4.3 Gaussian White Noise Excitation

4.3.1 General Remarks

In this section the method of equivalent linearization will be used for the study of the non-stationary amplitude response of a non-linear oscillator which is excited by a Gaussian white process.

The oscillator is assumed to be lightly damped and weakly non-linear. The level of excitation is also assumed to be relatively small. It was mentioned in the Introduction of this chapter that under the above assumptions the response of the system exhibits pseudo-sinusoidal behavior. The amplitude a and the phase θ of the response are therefore slowly varying random processes of the independent variables of the problem.

Knowledge of the amplitude of response of a certain system provides an envelope for the actual response. This knowledge is important for any design procedure which is probabilistically oriented. Currently, there is no sytematic method for the analysis of non-stationary amplitude response of non-linear second order dynamical systems. The existing methods cover only the area of the stationary amplitude response. Typically for this purpose, the variance of the actual response is calculated by any of the existing techniques [6] and subsequently the density of the amplitude distribution is approximated by Rayleigh distribution.

4.3.2 Formulation

For the purpose of the following discussion it will be assumed that the response of the non-linear system is described by the equation

$$m\ddot{x} + c\dot{x} + kx + \epsilon f(x, \dot{x}) = w(t) \quad . \tag{4.23}$$

The symbol w(t) stands for a Gaussian white process with spectral density S constant over all the frequency domain from minus

infinity ($-\infty$) to plus infinity ($+\infty$). The autocorrelation function of $w(t)$ is given by the formula

$$R(\tau) = E[w(t)w(t+\tau)] = 2\pi S\delta(\tau) \tag{4.24}$$

where E represents the expectation operator and $\delta(\tau)$ is the Dirac delta function. It is assumed that the quantity $|\epsilon f(x,\dot{x})|$ is small compared to $c\dot{x} + kx$ for all x and $\dot{x}$. The last condition may be interpreted as requiring that the solution of the non-linear problem differs only slightly from that of the linear problem obtained by neglecting all non-linear terms. The following additional assumptions about the damping of the system, and the spectral density S of the excitation are made

$$\frac{c}{m} = 0(\epsilon) \tag{4.25}$$

and

$$S = 0(\epsilon) \ . \tag{4.26}$$

In view of the smallness of the spectral density of the excitation, the damping coefficient and the non-linear term, the oscillation during a single cycle will be nearly harmonic. Hence,

$$x = a \cos(\omega_n t + \theta) \tag{4.27}$$

and

$$\frac{dx}{dt} = -a\,\omega_n \sin(\omega_n t + \theta) \ , \tag{4.28}$$

where a and θ are slowly varying functions of the variable t. The

symbol ω_n stands for the natural angular frequency of the non-linear system (4.23).

If equations (4.27) and (4.28) are solved, considering as unknown quantities the variables a and θ, the following relations are obtained

$$a^2 = x^2 + \frac{\dot{x}^2}{\omega_n^2} \tag{4.29}$$

and

$$\theta = \tan^{-1}\left(\frac{\dot{x}}{\omega_n x}\right) - \omega_n t \ . \tag{4.30}$$

The set of equations (4.29) and (4.30) provides another way to check the assumption of slowly varying amplitude a and phase θ. By analogy to the discussion given in section (4.2) for the case of harmonic excitation, herein, it is claimed that since the damping coefficient, the non-linearity parameter and the spectral density of the exciting random process are small, the rates of change of the energy of the system and the phase difference between the solution and the natural phase $\omega_n t$ are also small. Therefore, the amplitude a and the phase θ are slowly varying functions of the independent variable t.

It must be noticed that the effect of the non-linear function $\epsilon f(x, \dot{x})$ on θ has implicitly been considered through the change of the angular frequency of the system. Specifically, the natural angular frequency ω_n of the non-linear system (4.23) has been assumed to be different from the angular frequency

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{4.31}$$

of its linear part.

According to the preceding analysis, the amplitude a and the phase θ of the approximate solution of problem (4.23) can be assumed to be almost constant during one cycle of oscillation. The solution x and its derivative $\dot{x}$ are given by (4.27) and (4.28). Since the form of the approximate solution is known, the construction of an equivalent linear system for the non-linear system is a straightforward matter. The non-linear function $\epsilon f(x, \dot{x})$ can be replaced according to the rule [14]

$$\epsilon f(x, \dot{x}) \rightarrow \epsilon c_e \dot{x} + \epsilon k_e x \ , \tag{4.32}$$

where

$$c_e = - \frac{1}{\omega_n \pi a} \int_0^{2\pi} f(a \cos \psi, \ -a\omega_n \sin \psi) \sin \psi \, d\psi \tag{4.33}$$

and

$$k_e = \frac{1}{\pi a} \int_0^{2\pi} f(a \cos \psi, \ -a\omega_n \sin \psi) \cos \psi \, d\psi \ . \tag{4.34}$$

Therefore, the equivalent linear system of (4.23) is

$$m\ddot{x} + (c + \epsilon c_e)\dot{x} + (k + \epsilon k_e)x = w(t) \ . \tag{4.35}$$

The natural angular frequency of system (4.35) is given by the relation

$$\omega_n^2 = \frac{k + \epsilon k_e(a)}{m} \quad . \tag{4.36}$$

The expression given by equation (4.36) is also defined as the natural angular frequency of the non-linear system (4.23).

The solution of the non-linear system (4.23) and the equivalent linear system (4.35) as well is approximated by (4.27). The equivalent linear system (4.35) being derived, further analysis of the solution x can be made by using all the techniques which are applicable to linear systems excited by Gaussian white noise.

In Appendix A analysis of the amplitude response of a second order weakly damped linear dynamical system is made. The dynamical system is excited by a stationary wide-band random process, the spectral density of which is of order of the ratio of critical damping of the linear system. The major result of Appendix A is that the equation governing the amplitude of the response is not coupled with the phase. By applying this result to the equivalent linear system (4.35) the following equation is derived for the amplitude a of the solution (4.27).

$$\dot{a} = -\frac{c + \epsilon c_e(a)}{2m} a + \frac{\pi S}{2am^2\omega_n^2} + \frac{(\pi S)^{\frac{1}{2}}}{\omega_n m} \eta(t) \quad , \tag{4.37}$$

where $\eta(t)$ is a delta correlated process with zero mean

$$E[\eta(t)] = 0 \tag{4.38}$$

and

$$E[\eta(t)\eta(t + \tau)] = \delta(\tau) \quad . \tag{4.39}$$

It is noticed that equation (4.37) is of first order. Therefore, one immediate consequence of the present analysis is the reduction of the order of the equation which must be solved to obtain adequate information about the response of the non-linear system (4.23). Of course, equation (4.37) describes only the amplitude and not the phase of the response. This limitation is of minor importance however, since the amplitude of the response is what is most significant for almost all technical applications.

4.3.3 Fokker-Planck Equation

At the very beginning of the preceding analysis certain assumptions about the damping of the non-linear system and the spectral density of the excitation were made. These assumptions besides being vital for the derivation of (4.37) are important for the nature of the random process a(t). Specifically because of assumptions (4.25) and (4.26), equation (4.37) can be written as

$$\dot{a} = \varepsilon \Gamma[a, \eta(t)] \tag{4.40}$$

where $\varepsilon \Gamma[a, \eta(t)]$ represents the right hand side of (4.37).

In reference [50] it is proved that if $\eta(t)$ is a delta correlated random process and ε is a small parameter, the random process a(t) governed by the differential equation (4.37) can be regarded as Markovian. Therefore, its transition probability density function is governed by the Fokker-Planck equation which is associated with equation (4.37).

It is well known [51] that if only the linear part of the nonlinear system (4.23) is considered, the standard deviation σ of the stationary response can be estimated to be

$$\sigma^2 = \frac{\pi S}{2\zeta\omega_0^3 m^2} \tag{4.41}$$

where ω_0 is given by (4.31) and ζ is defined by

$$\frac{c}{m} = 2\zeta\omega_0 \quad . \tag{4.42}$$

Using equation (4.41), equation (4.37) can be rewritten as

$$\dot{a} = -\zeta\omega_0\left(a - \frac{\sigma^2}{a}\frac{\omega_0^2}{\omega_n^2}\right) - \epsilon\frac{c_e(a)}{2m}a + (2\zeta\omega_0\sigma^2)^{\frac{1}{2}}\frac{\omega_0}{\omega_n}\eta(t) \quad . \tag{4.43}$$

The Fokker-Planck equation which is associated with (4.43) is [26]

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left\{p\left[\zeta\omega_0\left(a - \frac{\sigma^2}{a}\frac{\omega_0^2}{\omega_n^2}\right) + \epsilon\frac{c_e(a)}{2m}a\right]\right\}$$

$$+ \frac{\partial}{\partial a}\left\{\zeta\omega_0\sigma^2\left(\frac{\omega_0}{\omega_n}\right)^2\frac{\partial p}{\partial a} + \tfrac{1}{2}\zeta\omega_0\sigma^2 p\frac{\partial}{\partial a}\left(\frac{\omega_0}{\omega_n}\right)^2\right\} \quad , \tag{4.44}$$

where $p(a,t)$ is the probability density describing $a(t)$. If the nonlinearity parameter ϵ is equal to zero, the quantity $\epsilon c_e(a)$ equals zero and the ratio ω_0/ω_n equals unity. Therefore, for $\epsilon = 0$ equation (4.44) can be written as

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left[\zeta\omega_0\left(a - \frac{\sigma^2}{a}\right)p\right] + \zeta\omega_0\sigma^2\frac{\partial^2 p}{\partial a^2} \quad . \tag{4.45}$$

Further analysis of (4.44) and (4.45) requires specification of boundary and initial conditions for a. Since a represents the amplitude of the response x(t), it cannot take negative values. Therefore, its boundaries are defined by

$$0 \leq a \leq \infty \quad . \tag{4.46}$$

If the above boundary conditions are imposed, equation (4.45) has eigenvalues given by [52]

$$\lambda_n = 2n\zeta\omega_0 \tag{4.47}$$

and eigenfunctions proportional to [52]

$$A_n(a) = \frac{1}{n!}\frac{ae^{-a^2/2\sigma^2}}{\sigma^2}L_n\left(\frac{a^2}{2\sigma^2}\right) \tag{4.48}$$

where L_n is the Laguerre polynomial of order n. The complete solution of (4.45) is therefore given by

$$p(a,t) = \sum_{n=0}^{\infty} T_n A_n(a) e^{-\lambda_n t} \quad . \tag{4.49}$$

The coefficients T_n must be calculated using the initial condition. It is assumed that the system is initially (t = 0) at rest. Probabilistically speaking, this means that a equals zero at t = 0 with probability 1. In terms of the probability density the last

condition is written as

$$p(a,0) = \delta(a) \tag{4.50}$$

where the symbol $\delta(a)$ stands for the one sided Dirac delta function. Because of condition (4.50) it can be proved easily that

$$T_n = 1 \; ; \qquad n = 0, \ldots \; . \tag{4.51}$$

Therefore (4.49) can be rewritten as

$$p(a,t) = \frac{a}{\sigma^2} e^{-a^2/2\sigma^2} \sum_{n=0}^{\infty} \frac{1}{n!} L_n\left(\frac{a^2}{2\sigma^2}\right) e^{-2n\zeta\omega_0 t} \; . \tag{4.51a}$$

It is evident that

$$\lim_{t \to \infty} p(a,t) = \frac{a}{\sigma^2} e^{-a^2/2\sigma^2} \tag{4.52}$$

which is the required process describing the probability distribution of the amplitude of a stationary Gaussian process. It is felt that this provides a certain measure of the applicability of the method used.

Since the solution of equation (4.44) for ϵ equal zero is readily available, it appears that perturbation techniques provide a convenient vehicle for the solution of equation (4.44). In reference [53] this method was used extensively for solving the Fokker-Planck equation associated with first-order stochastic differential equations. Before any further work toward this end, some simplification of the non-linear expression (4.36) for ω_n is necessary.

Using equation (4.31), equation (4.36) can be rewritten as

$$\omega_n^2 = \omega_0^2 + \epsilon \frac{k_e(a)}{m} \tag{4.53}$$

or putting

$$\omega_e^2(a) = \frac{k_e(a)}{m} \tag{4.54}$$

equation (4.53) can be written as

$$\omega_n^2 = \omega_0^2 + \epsilon\omega_e^2(a) \quad . \tag{4.55}$$

It was crucial for the analysis of part 4.3.2 that the contribution of the non-linear term $\epsilon f(x, \dot{x})$ to the response of the system (4.23) be small. Since a is a random process, it is clear that this condition should be interpreted stochastically. The amplitude can take any value between zero and infinity with finite probability. The amplitude probability density of the linear part of the system (4.23) is a delta function at $t = 0$ and spreads to a Rayleigh distribution at $t = \infty$. From Figure 4.1 it is seen that for a Rayleigh distribution the most probable value of a is σ, the mean value of a is $\sqrt{\pi/2}\,\sigma$, and the probability that a takes a value greater than 3σ is equal to 0.0111. All the above statistical parameters show that the major contribution of the amplitude probability distribution corresponds to ordinates from zero up to 3σ. With this guideline from the linear oscillator, equation (4.44) will be solved for those values of a for which the assumption

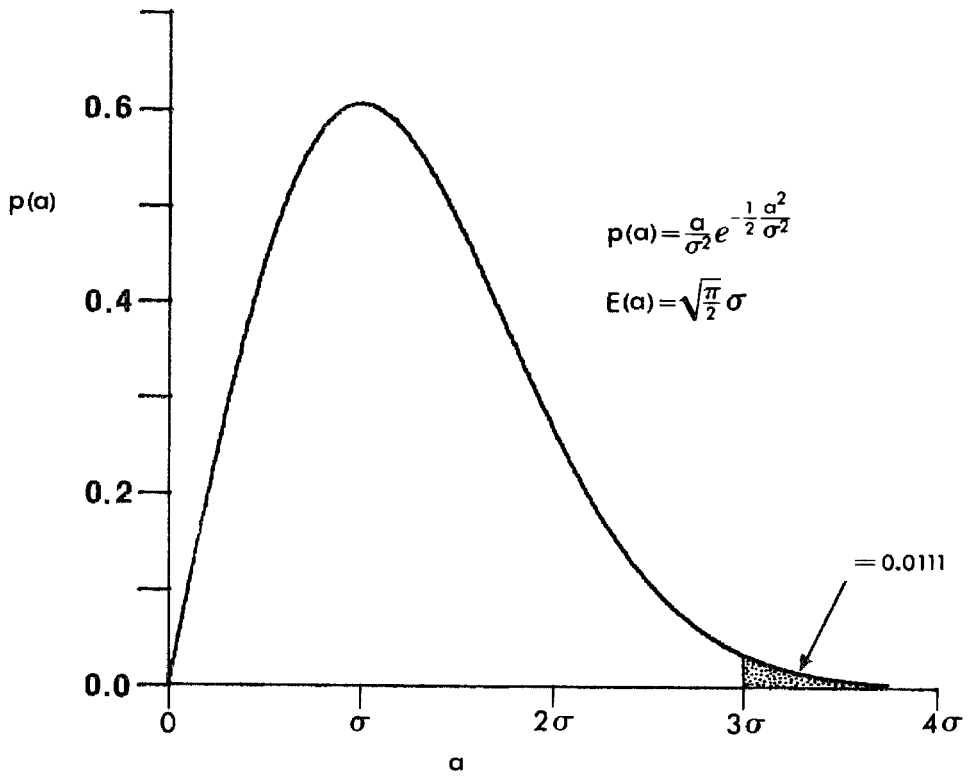


Figure 4.1. Rayleigh Probability Density Function

$$\epsilon\omega_e^2(a) \ll 1 \tag{4.56}$$

is valid.

Because of assumption (4.56), equation (4.55) implies that

$$\frac{\omega_0^2}{\omega_n^2} = 1 - \epsilon \frac{\omega_e^2(a)}{\omega_0^2} . \tag{4.57}$$

Equation (4.57) is correct up to $0(\epsilon)$. Using equation (4.57), equation (4.44) can be rewritten

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left[\zeta\omega_0\left(a - \frac{\sigma^2}{a}\right)p\right] + \zeta\omega_0\sigma^2 \frac{\partial^2 p}{\partial a}$$

$$+ \epsilon\left\{\frac{\partial}{\partial a}\left[\frac{\zeta\sigma^2\omega_e^2(a)}{\omega_0}\left(\frac{p}{a} - \frac{\partial p}{\partial a}\right) + \frac{pc_e(a)a}{2m} - \frac{1}{2}\frac{\zeta\sigma^2 p}{\omega_0}\frac{d}{da}[\omega_e^2(a)]\right]\right\} \tag{4.58}$$

Equation (4.58) will be solved by using the technique of separation of variables. A solution of the form

$$p(a,t) = A_{NL}(a)e^{-\lambda_{NL}t} \tag{4.59}$$

is sought. Substituting expression (4.59) in equation (4.58) the following equation is obtained

$$\zeta\omega_0\sigma^2 \frac{d^2A_{NL}}{da^2} + \zeta\omega_0 \frac{d}{da}\left[\left(a - \frac{\sigma^2}{a}\right)A_{NL}\right] + \lambda_{NL}A_{NL} = \epsilon H_{NL} \tag{4.60}$$

where H_{NL} is given by

$$H_{NL} = -\frac{d}{da}\left[\frac{\zeta\sigma^2\omega_e^2(a)}{\omega_0}\left(\frac{A_{NL}}{a} - \frac{dA_{NL}}{da}\right) + \frac{A_{NL}c_e(a)}{2m} - \frac{\zeta\sigma^2 A_{NL}}{2\omega_0}\frac{d}{da}[\omega_e^2(a)]\right] . \tag{4.61}$$

If ϵ equals zero, equation (4.60) has eigenvalues and eigenvectors given by (4.47) and (4.48) respectively. For the determination of the eigenvalues and eigenfunctions of equation (4.60) for non-zero value of ϵ, the classical perturbation technique which is discussed in [54] will be used.

An asymptotic expansion to $0(\epsilon)$ for the eigenfunctions and eigenvalues of equation (4.60) will be obtained by assuming expansions of the form

$$A_{NL,m} = A_m + \epsilon B_m \tag{4.62}$$

and

$$m = 0, \ldots, n$$

$$\lambda_{NL,m} = \lambda_m + \epsilon\rho_m \tag{4.63}$$

where A_m and λ_m are given by equations (4.48) and (4.47) respectively. Substituting expressions (4.62) and (4.63) into the differential equation (4.60), collecting terms with like powers of ϵ, and equating each group to zero, the following set of equations is derived

$$0(\epsilon^0) : \zeta\omega_0\sigma^2\frac{d^2A_m}{da^2} + \zeta\omega_0\frac{d}{da}\left(aA_m - \sigma^2\frac{A_m}{a}\right) + \lambda_m A_m = 0 \tag{4.64}$$

$$0(\epsilon^1) : \zeta\omega_0\sigma^2\frac{d^2B_m}{da^2} + \zeta\omega_0\frac{d}{da}\left(aB_m - \frac{\sigma^2 B_m}{a}\right) + \lambda_m B_m = -\rho_m A_m + H_m \tag{4.65}$$

where H_m is given by the relation

$$H_m = -\frac{d}{da}\left[\frac{\zeta\sigma^2\omega_e^2(a)}{\omega_0}\left(\frac{A_m}{a} - \frac{dA_m}{da}\right) + \frac{A_m c_e(a)}{2m} \quad \frac{\zeta\sigma^2 A_m}{2\omega_0}\frac{d}{da}[\omega_e^2(a)]\right] \quad (4.66)$$

Equations (4.64) and (4.63) may be used for the determination of B_m and ρ_m. According to Appendix B, the following orthonormality condition is true.

$$\int_0^\infty \frac{A_m A_n}{A_0}\, da = \delta_{mn} \quad , \quad (4.67)$$

where δ_{mn} is the Kronecker delta symbol. In other words the set of functions A_m; m = 0, . . . , constitute an orthonormal basis in terms of which the function B_m can be expressed as

$$B_m = \sum_{k=0}^{\infty} \beta_{m,k} A_k \quad , \quad (4.68)$$

where $\beta_{m,k}$ are coefficients determined by the following relation

$$\beta_{m,k} = \int_0^\infty B_m A_k da \quad ; \quad \begin{matrix} m = 0, 1, \ldots \\ k = 0, 1, \ldots \end{matrix} \quad . \quad (4.69)$$

Substituting expression (4.68) into (4.65) and using relation (4.64) for m = 0, 1, . . . the following relation is derived

$$\sum_{k=0}^{\infty} \beta_{m,k} A_k (\lambda_m - \lambda_k) = -\rho_m A_m + H_m \quad . \quad (4.70)$$

Multiplying both sides of equation (4.70) by A_k/A_0, integrating from zero (0) to infinity (∞) and using the orthonormality condition (4.67) gives

$$\beta_{m,k}(\lambda_m - \lambda_k) = -\rho_m \int_0^\infty \frac{A_m A_k}{A_0}\, da + \int_0^\infty \frac{H_m A_k}{A_0}\, da \quad . \tag{4.71}$$

Applying equation (4.71) for $m = k$ yields

$$\rho_m = \int_0^\infty \frac{H_m A_m}{A_0}\, da \quad . \tag{4.72}$$

If $k \neq m$, equation (4.71) may be rewritten as

$$\beta_{m,k} = \frac{1}{\lambda_m - \lambda_k} \int_0^\infty \frac{H_m A_k}{A_0}\, da \quad . \tag{4.72a}$$

It is clear that the coefficient β_{mm} cannot be determined from equation (4.72). In fact, it is easily seen by direct substitution in (4.65) that any finite value for β_{mm} will be acceptable for the perturbation analysis. This should be expected, because by the perturbation analysis essentially the non-linear eigenfunction $A_{NL,m}$ is expressed as a linear combination of the eigenfunctions A_m; $m = 0, \ldots$. Clearly the product of $A_{NL,m}$ with any constant number is again an eigenfunction of order m. Therefore, even if the coefficients $\beta_{m,k}$ for $m \neq k$ are determined, the coefficient $1 + \epsilon\beta_{mm}$ of A_m is indeterminate.

The complete solution of the Fokker-Planck equation (4.58) is given by the expression

$$p_{NL}(a,t) = \sum_{k=0}^{\infty} T_{NL,k} e^{-\lambda_{NL,k} t} A_{NL,k}(a) \qquad (4.73)$$

where $T_{NL,k}$ are coefficients to be determined by the initial condition (4.50). It is noticed that condition (4.50) does not contain any function of ϵ. Similarly to the case of typical perturbation problems [55], herein the initial condition (4.50) will be satisfied by the zeroth order terms of expansion (4.73). Subsequently, the undetermined coefficients β_{mm} of the first order in ϵ terms will be determined so that the overall condition (4.50) is satisfied.

Using equation (4.73), the initial condition (4.50) can be rewritten as

$$\delta(a) = \sum_{k=0}^{\infty} T_{NL,k} A_{NL,k}(a) \ . \qquad (4.74)$$

Multiplying both sides of (4.74) by A_k/A_0 and integrating from zero (0) to infinity (∞) yields

$$\frac{A_k(0)}{A_0(0)} = T_{NL,k} + \epsilon \sum_{r=0}^{\infty} T_{NL,r} \beta_{r,k} \ . \qquad (4.75)$$

For the derivation of (4.75) the orthonormality property (4.67) has been used. By using formula (4.28) for $A_k(a)$ it is easily seen that

$$\frac{A_k(0)}{A_0(0)} = 1 \; ; \qquad k = 0, \ldots \; . \tag{4.76}$$

Therefore, equation (4.75) can be rewritten as

$$1 = T_{NL,k} + \epsilon \sum_{r=0}^{\infty} T_{NL,r} \beta_{r,k} \; . \tag{4.77}$$

According to the preceding analysis, condition (4.50) will be satisfied by requiring that

$$T_{NL,k} = 1 \; ; \qquad k = 0, \ldots \tag{4.78}$$

and

$$\sum_{r=0}^{\infty} \beta_{r,k} = 0 \; ; \qquad k = 0, \ldots \; . \tag{4.79}$$

Equation (4.79) can be rewritten as

$$\beta_{mm} = - \sum_{r \neq m}^{\infty} \beta_{r,m} \; ; \qquad m = 0, \ldots \; . \tag{4.80}$$

At this point all the steps of the solution of Fokker-Planck equation (4.4) by the perturbation techniques have been completed.

It is worth noticing that the stationary solution of (4.44) can be found without using any approximate technique. Let

$$\Lambda(a) = -\zeta\omega_0 \left(a - \frac{\sigma^2}{a} \frac{\omega_0^2}{\omega_n^2} \right) - \frac{\epsilon c_e(a)}{2m} a \tag{4.81}$$

and

$$\Delta(a) = 2\zeta\omega_0\sigma^2\left(\frac{\omega_0}{\omega_n}\right)^2 \quad . \tag{4.82}$$

Thus, equation (4.44) can be written as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial a}[\Lambda p] + \tfrac{1}{4}\frac{\partial}{\partial a}\left[\Delta\frac{\partial p}{\partial a} + \frac{\partial}{\partial a}(\Delta p)\right] \quad . \tag{4.83}$$

The stationary solution p_s of (4.83) satisfies the differential equation

$$-\frac{\partial}{\partial a}[\Lambda p_s] + \tfrac{1}{4}\frac{\partial}{\partial a}\left[\Delta\frac{\partial p_s}{\partial a} + \frac{\partial}{\partial a}(\Delta p_s)\right] = 0 \quad . \tag{4.84}$$

Equation (4.84) is directly integrable. The solution p_s is given by the formula [26]

$$p_s(a) = \frac{\theta}{\sqrt{\Delta}}\exp\left\{2\int\frac{\Lambda}{\Delta}\,da\right\} \tag{4.85}$$

where θ is a constant determined by the condition

$$\int_0^\infty p_s(a)da = 1 \quad . \tag{4.86}$$

Obviously, the solution (4.85) should agree up to $0(\epsilon)$ with $A_{NL,0}$ which represents the stationary part of the non-stationary probability distribution $p_{NL}(a,t)$ derived by perturbation techniques. This requirement is checked in the specific example problems which are presented in a subsequent chapter of this thesis.

From a practical point of view, there is a question of how to calculate the integrals in equations (4.72) and (4.72a) which are necessary for the determination of ρ_m and $\beta_{m,k}$. Besides the answer of resorting to numerical calculation by digital computer, it was felt that certain analytic work should be done toward this goal. The results of this study are summarized in Appendix B. There, analytic formulae for determining the values of integrals of the forms $\int_0^\infty a^{2s} \frac{A_n A_k}{A_0} da$, $\int_0^\infty a A_k da$ and $\int_0^\infty a^2 A_k da$ are presented. Iterative schemes for evaluating $A_n(a)$ and $dA_n(a)/da$ are also presented. Repeated usage of the formulae of Appendix B will be made in the solution of specific example problems which will follow.

4.3.4 Summary

The method of equivalent linearization has been utilized for the derivation of a first order stochastic differential equation which governs the amplitude of the non-stationary response of a system with small non-linearity. The system is asymptotic to a linear lightly damped oscillator with Gaussian white noise excitation. The fact that the differential equation which governs the response amplitude is of first order, makes the problem amenable to the techniques which are available for first-order non-linear stochastic differential equations.

Since it has been assumed that the non-linearity, the damping, and the spectral density of the excitation are small and that the

excitation is delta correlated, an approximation of the amplitude of response by a Markovian process is justified. Subsequently the Fokker-Planck equation for the probability density of the amplitude is considered. The stationary solution of this equation is readily found by existing formulae. The governing Fokker-Planck equation is asymptotic to a partial differential equation, the eigenvalues and eigenfunctions of which are known analytically. Therefore, perturbation techniques are used to obtain an approximation of the non-stationary distribution of the amplitude of the response. In accordance with this approach, the coefficients of an accurate, up to $0(\epsilon)$, series representation of the eigenvalues and the eigenfunctions of the non-linear problem are determined. The stationary probability density distribution of the amplitude is found by direct integration of the time independent part of the corresponding Fokker-Planck equation.

V. Example Studies

5.1 Example of Duffing Oscillator

5.1.1 General Remarks

The objective of this example study is to utilize the general method developed in section (4.2) for the examination of the non-stationary random response of a lightly damped hardening Duffing oscillator. This specific type of non-linearity is selected because many interesting non-linear problems can be modeled by the cubic non-linearity. The system is excited by a Gaussian white process. Approximate expressions for the statistics of the non-stationary amplitude response of the system will be given. Comparison of the theoretical analysis with the results obtained by Monte Carlo simulation will be made.

All of the symbols used in the following analysis represent the same physical quantities as for the general analysis given in Chapter IV.

5.1.2 Formulation

Consider a Duffing oscillator excited by a Gaussian white process. The differential equation which governs the response of the system is

$$m\ddot{Y} + c\dot{Y} + kY\left(1 + \epsilon \frac{Y^2}{\sigma^2}\right) = w(t) \quad . \tag{5.1}$$

If the small non-linear parameter ϵ is zero, the standard deviation σ of the stationary response is given by equation (4.41). For the purpose of obtaining results in terms of dimensionless parameters,

the following transformation is introduced.

$$x = \frac{\Psi}{\sigma} \quad . \tag{5.2}$$

Using transformation (5.2) the non-linear dynamical system (5.1) can be rewritten as

$$m\ddot{x} + c\dot{x} + kx(1 + \epsilon x^2) = \frac{w(t)}{\sigma} \quad . \tag{5.3}$$

Clearly, system (5.3) belongs to the general class of systems represented by equation (4.23). Therefore, the analysis which was performed in section (4.2) for the case of the general problem (4.23) is applicable herein as well. For the present case the non-linear function $f(x, \dot{x})$ is

$$f(x, \dot{x}) = kx^3 \tag{5.4}$$

and the spectral density of the excitation is replaced by

$$S \rightarrow \frac{S}{\sigma^2} \quad . \tag{5.5}$$

The solution x and its derivative dx/dt are given in terms of the amplitude a and the phase θ by equations (4.27) and (4.28).

The equivalent stiffness of the system may be determined by formula (4.34). This gives

$$k_e = \frac{1}{\pi a} \int_0^{2\pi} ka^3 \cos^4 \psi \, d\psi = \frac{3k}{4} a^2 \quad . \tag{5.6}$$

The equivalent linear system for the non-linear system (5.3) therefore becomes

$$m\ddot{x} + c\dot{x} + k\left(1 + \tfrac{3}{4}\epsilon a^2\right) x = \frac{w(t)}{\sigma} \quad . \tag{5.7}$$

The natural angular frequency of the equivalent linear system (5.7) is

$$\omega_n^2 = \omega_0^2 \left(1 + \tfrac{3}{4}\epsilon a^2\right) \quad . \tag{5.8}$$

The stochastic differential equation governing the amplitude of the response of (5.3) is given by the general formula (4.37). This gives

$$\dot{a} = -\frac{c}{2m} a + \frac{\pi S}{2am^2\omega_n^2\sigma^2} + \frac{(\pi S)^{\frac{1}{2}}}{\omega_n m\sigma} \eta(t) \quad , \tag{5.9}$$

where $\eta(t)$ is a delta correlated process with zero mean. Using the ratio of critical damping ζ, equation (5.9) can be rewritten as

$$\begin{aligned} \dot{a} &= -\zeta\omega_0 \left(a - \frac{1}{a}\frac{\omega_0^2}{\omega_n^2}\right) + (2\zeta\omega_0)^{\frac{1}{2}} \frac{\omega_0}{\omega_n} \eta(t) \\ &= -\zeta\omega_0 \left(a - \frac{1}{a\left(1 + \frac{3}{4}\epsilon a^2\right)}\right) + \left(\frac{2\zeta\omega_0}{1 + \frac{3}{4}\epsilon a^2}\right)^{\frac{1}{2}} \eta(t) \quad . \end{aligned} \tag{5.10}$$

5.1.3 Fokker-Planck Equation

The Fokker-Planck equation which is associated with equation (5.10) is given by formula (4.44) as

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left\{p\left[\zeta\omega_0\left(a - \frac{1}{a\,(1+\frac{3}{4}\epsilon a^2)}\right)\right]\right\}$$

$$+ \frac{\partial}{\partial a}\left\{\frac{\zeta\omega_0}{1+\frac{3}{4}\epsilon a^2}\frac{\partial p}{\partial a} + \frac{1}{2}\zeta\omega_0 p\frac{\partial}{\partial a}\left(\frac{1}{1+\frac{3}{4}\epsilon a^2}\right)\right\} \quad . \qquad (5.11)$$

A first order non-stationary solution of (5.11) will be next determined. The derived solution will be valid for those values of a such that

$$\frac{3}{4}\epsilon a^2 \ll 1 \quad . \qquad (5.12)$$

Because of assumption (5.12), the term $1/(1+\frac{3}{4}\epsilon a^2)$ can be rewritten as in equation (4.57). Namely,

$$\frac{1}{1+\frac{3}{4}\epsilon a^2} = 1 - \frac{3}{4}\epsilon a^2 \quad . \qquad (5.13)$$

Equation (5.13) is correct up to $0(\epsilon)$. Exploiting approximation (5.13) the Fokker-Planck equation (5.11) can be written as

$$\frac{\partial p}{\partial t} = \zeta\omega_0\left[\frac{\partial^2 p}{\partial a^2} + \frac{\partial}{\partial a}(ap) - \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right] + \frac{3}{4}\zeta\omega_0\epsilon\left[\frac{\partial(ap)}{\partial a} - \frac{\partial}{\partial a}\left(a^2\frac{\partial p}{\partial a}\right) - \frac{\partial(ap)}{\partial a}\right]$$

$$= \zeta\omega_0\left[\frac{\partial^2 p}{\partial a^2} + \frac{\partial}{\partial a}(ap) - \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right] - \frac{3}{4}\zeta\omega_0\epsilon\frac{\partial}{\partial a}\left(\frac{a^2\partial p}{\partial a}\right) \quad . \qquad (5.14)$$

For $\epsilon = 0$, equation (5.14) becomes

$$\frac{\partial p}{\partial t} = \zeta\omega_0\left[\frac{\partial^2 p}{\partial a^2} + \frac{\partial}{\partial a}(ap) - \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right] \quad . \tag{5.15}$$

It is assumed that the same boundary and initial conditions hold for the present example as they did for the general case. The eigenvalues and the eigenfunctions of equation (5.15) are given by equations (4.47) and (4.48) for $\sigma = 1$.

Following the general analysis performed in section 3.2 a first order solution of equation (5.15) may be found. For this purpose the general formulae of section 3.2 will be used. The ultimate goal is to calculate the quantities ρ_m and $\beta_{n,k}$ of the perturbation solution which are given by formulae (4.72), (4.72a), and (4.80). Observing equations (4.72) and (4.72a) it is understood that the expression H_m defined by equation (4.66) must be determined so that coefficients ρ_m and $\beta_{m,k}$ can be calculated. Considering equation (5.14) it is easily verified that

$$H_m = \tfrac{3}{4}\zeta\omega_0 \frac{d}{da}\left(a^2 \frac{dA_m}{da}\right) \quad ; \qquad m = 0, \ldots \quad . \tag{5.16}$$

Using formula (B31) of Appendix B equation (5.16) can be rewritten as

$$\begin{aligned} H_m &= \tfrac{3}{4}\zeta\omega_0 \frac{d}{da}\{a[2(m+1)A_{m+1} - (2m+1)A_m]\} \\ &= \tfrac{3}{4}\zeta\omega_0\left[2(m+1)A_{m+1} - (2m+1)A_m + 2(m+1)\frac{dA_{m+1}}{da} - (2m+1)\frac{dA_m}{da}\right] \quad . \end{aligned} \tag{5.17}$$

Again, applying formula (B31), equation (5.17) can be written as

$$H_m = \frac{3}{2}\zeta\omega_0[2(m+1)(m+2)A_{m+2} - (m+1)(4m+3)A_{m+1} + m(2m+1)A_m] \quad . \tag{5.18}$$

Using the general formula (4.72) and the orthonormality relation (B8), it is found that

$$\rho_m = \int_0^\infty \frac{H_m A_m}{A_0}\, da = \frac{3}{2}\zeta\omega_0 m(2m+1) \quad ; \qquad m = 0, 1, \ldots \quad . \tag{5.19}$$

Using the general formula (4.72a) and the orthonormality relation (B8), it may further be shown that

$$\beta_{m,m+1} = \frac{1}{\lambda_m - \lambda_{m+1}} \int_0^\infty \frac{H_m A_{m+1}}{A_0}\, da = \tfrac{3}{4}(m+1)(4m+3) \quad ; \qquad m = 0, \ldots \quad . \tag{5.20}$$

$$\beta_{m,m+2} = \frac{1}{\lambda_m - \lambda_{m+2}} \int_0^\infty \frac{H_m A_{m+2}}{A_0}\, da = -\tfrac{3}{2}(m+1)(m+2) \quad ; \qquad m = 0, \ldots \quad . \tag{5.21}$$

$$\beta_{m,k} = 0 \quad ; \qquad k \neq m+ \, , m+1, m \tag{5.22}$$

Using formula (4.80) gives

$$\begin{aligned} \beta_{m,m} &= -(\beta_{m-2,m} + \beta_{m-1,m}) \\ &= -\{-\tfrac{3}{4} m(m-1) + \tfrac{3}{4} m[4(m-1)+3]\} \\ &= \frac{9}{4} m^2 \quad ; \qquad m = 0, \ldots \quad . \end{aligned} \tag{5.23}$$

The coefficients ρ_m and $\beta_{m,m}, \beta_{m,m+1}$ and $\beta_{m,m+2}$ being calculated, the first order solution of equation (5.14) is given by equation (4.73).

The stationary solution of equation (5.11) is given by the general formula (4.85). Applying this formula to the present case it is found that

$$p_s = \frac{\theta}{\sqrt{\Delta(a)}} \exp\left\{2\int \frac{\Lambda(a)}{\Delta(a)}\, da\right\}, \tag{5.24}$$

where

$$\Lambda(a) = -\zeta\omega_0\left[a - \frac{1}{a\,(1 + \frac{3}{4}\epsilon a^2)}\right] \tag{5.25}$$

and

$$\Delta(a) = \frac{2\zeta\omega_0}{1 + \frac{3}{4}\epsilon a^2}\,. \tag{5.26}$$

Substituting expressions (5.25) and (5.26) in equation (5.24) it is seen that

$$p_s(a) = \frac{a e^{-a^2/2}\sqrt{1 + \frac{3}{4}\epsilon a^2}\; e^{-\frac{3}{16}\epsilon a^4}}{\int_0^\infty a e^{-a^2/2}\sqrt{1 + \frac{3}{4}\epsilon a^2}\; e^{-\frac{3}{16}\epsilon a^4}\, da}\,. \tag{5.27}$$

If $\epsilon = 0$, formula (5.27) yields the classical Rayleigh probability distribution for the amplitude. From equation (5.27) it is seen that the non-linearity of the system acts to suppress the probability of big values of the amplitude a for $\epsilon > 0$.

It is interesting to compare the first order representation of (5.27) with the eigenfunction $A_{NL,0}$ corresponding to m = 0, determined by the perturbation technique. Using formulae (5.21), (5.22) and (5.23) it is found that

$$A_{NL,0} = A_0 + \varepsilon(\beta_{00}A_0 + \beta_{01}A_1 + \beta_{02}A_2) \tag{5.28}$$

where

$$\beta_{00} = 0 \tag{5.29}$$

$$\beta_{01} = \frac{9}{4} \tag{5.30}$$

and

$$\beta_{02} = -\frac{3}{2} \ . \tag{5.31}$$

Therefore, equation (5.28) can be rewritten as

$$A_{NL,0} = A_0 + \varepsilon(\frac{9}{4}A_1 - \frac{3}{2}A_2) \ . \tag{5.32}$$

Using the general formula (4.48) gives

$$A_0 = ae^{-a/2} \tag{5.33}$$

$$A_1 = ae^{-a^2/2}\left(-\frac{a^2}{2} + 1\right) \tag{5.34}$$

$$A_2 = \tfrac{1}{2}\, ae^{-a^2/2}\left(\frac{a^4}{4} - \frac{4a^2}{2} + 2\right) \ . \tag{5.35}$$

Substituting the above expressions for A_0, A_1 and A_2 in equation (5.32) yields

$$A_{NL,0} = ae^{-a^2/2}\left[1 - \frac{3\varepsilon}{16}(a^4 - 2a^2 - 4)\right] \ . \tag{5.36}$$

To first order in ϵ it may be seen that

$$e^{-\frac{3}{16}\epsilon a^4} = 1 - \frac{3}{16}\epsilon a^4 \tag{5.37}$$

and

$$\sqrt{1 + \tfrac{5}{4}\epsilon a^2} = 1 + \tfrac{5}{8}\epsilon a^2 \quad \text{for } (\epsilon a^2 < 1) \ . \tag{5.38}$$

Using these approximations in equation (5.27), it is found that

$$\int_0^\infty a e^{-a^2/2} \sqrt{1 + \tfrac{5}{4}\epsilon a^2}\, e^{-\frac{3}{16}\epsilon a^4} da$$

$$\approx \int_0^\infty a e^{-a^2/2} [1 - \frac{3}{16}\epsilon(a^4 - 2a^2)] da$$

$$= 1 - \tfrac{5}{4}\epsilon a \ . \tag{5.39}$$

Using equations (5.37), (5.38) and (5.39), equation (5.27) yields

$$p_s(a) = a e^{-a^2/2} [1 - \frac{3\epsilon}{16}(a^4 - 2a^2 - 4)] \ . \tag{5.40}$$

By equations (5.36) and (5.40) it is confirmed that the first order solutions for the stationary probability density function as determined either by perturbation techniques applied to the general equation (5.10) or by direct application of formula (5.24) are, as they should be, identical.

5.1.4 Numerical Calculations

Expression (4.73) is a first order in ϵ solution of equation (5.14). Therefore, strictly speaking, good agreement of (4.73) with

the exact solution of (5.14) should be expected only for small values of the non-linearity parameter ϵ. However, expression (5.27) is the exact stationary solution of equation (5.14); therefore good agreement of expression (5.27) with the actual probability distribution of the amplitude of the response of system (5.1) should be expected even for large values of ϵ.

It is important to note that the coefficient of the non-linear function in system (5.1) is divided by σ^2; therefore the actual non-linearity parameter of system (5.1) is the product $\epsilon\sigma^2$. This simply means that if the spectral density of the excitation is sufficiently small so that σ is small as well, the solution expressed by equation (4.73) is reliable even for significantly large values of ϵ.

The non-stationary solution represented by equation (4.73) was used for the computation of the time dependent mean value and standard deviation of the amplitude of the response of the non-linear system (5.1). Equation (5.27) was used for the determination of the steady-state values of the above statistical parameters.

By a careful inspection of equation (5.14) which must be solved for the determination of the non-stationary probability density function of the amplitude of the response of system (5.1), it is recognized that the ratio of critical damping ζ and the non-linearity coefficient ϵ can be used to identify the problem (5.1). Specifically, by the transformation represented by equation (5.2), the amplitude of the response of the system (5.1) is normalized by the standard deviation σ of the stationary response of the linear part of system (5.1). In this manner, the dependence of the actual

solution of system (5.1) on the spectral density S of the excitation the mass m, the damping c and the stiffness k is accounted for through σ. Furthermore, it is easily verified that the time dependence of the solution of equation (5.14) can be expressed in terms of the dimensionless time

$$t' = \frac{t}{T} \tag{5.41}$$

where

$$T = \frac{2\pi}{\omega_0} \, . \tag{5.42}$$

In this manner the explicit dependence of the solution of equation (5.14) on ω_0 is eliminated.

The value of the ratio of critical damping ζ of system (5.1) was taken herein to be:

$$\zeta = 0.02 \, . \tag{5.43}$$

Varicus values were assigned to the non-linearity parameter ε.

For the computation of the mean value E(a), the values of the integrals $\int_0^\infty aA_n da$; n = 0,..., were computed by the explicit iterative scheme represented by equation (B38) of Appendix B. For the computation of the standard deviation it was necessary to determine the values of the integrals $\int_0^\infty a^2 A_n$; n = 0,... . For this purpose, formulae (B43), (B44) and (B45) were used. Figures 5.1 to 5.6 show the computed time dependent mean value and

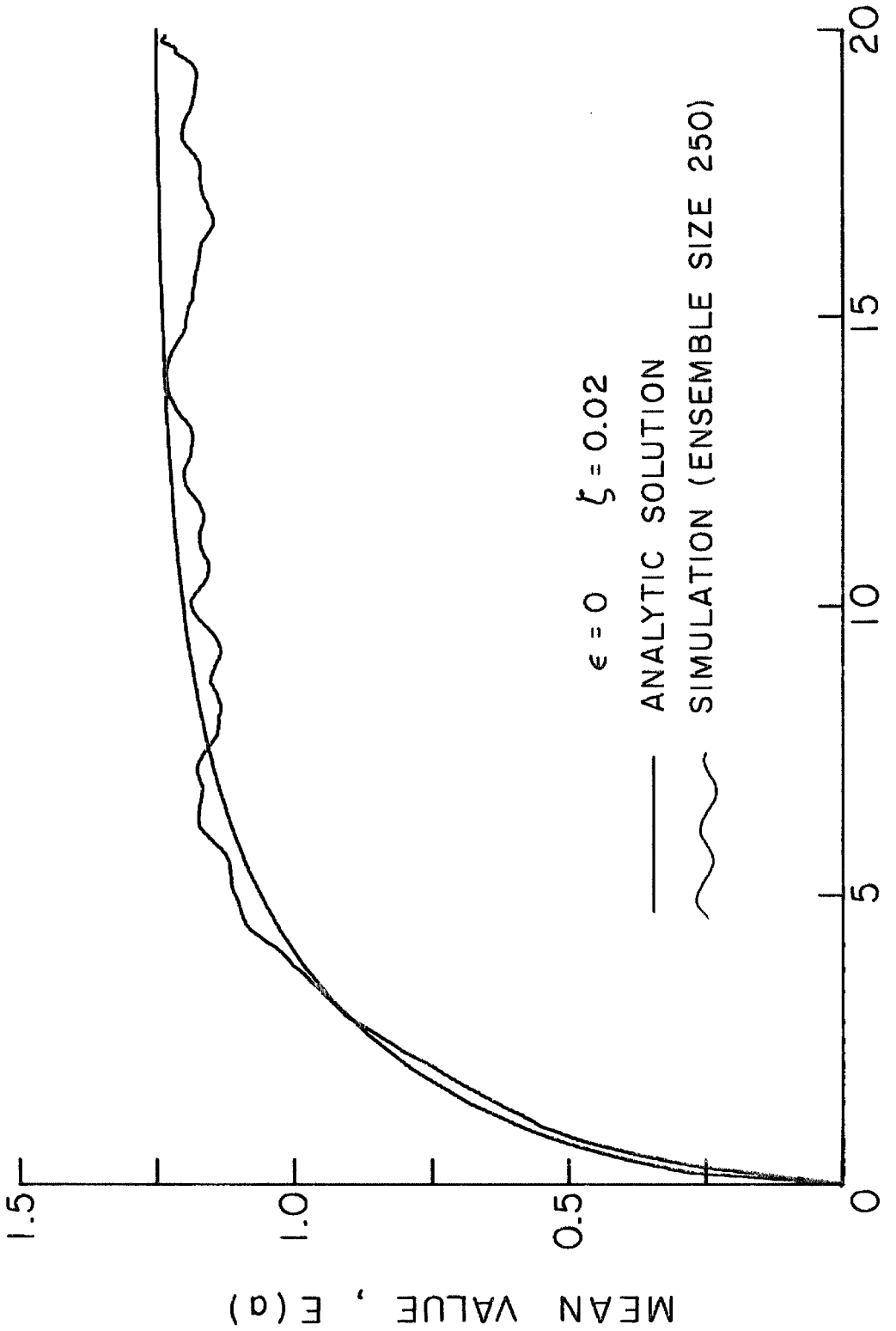


Figure 5.1. Mean Value of Non-Stationary Response Amplitude versus Time. Duffing Oscillator, $\epsilon = 0.0$ and $\zeta = 0.02$.

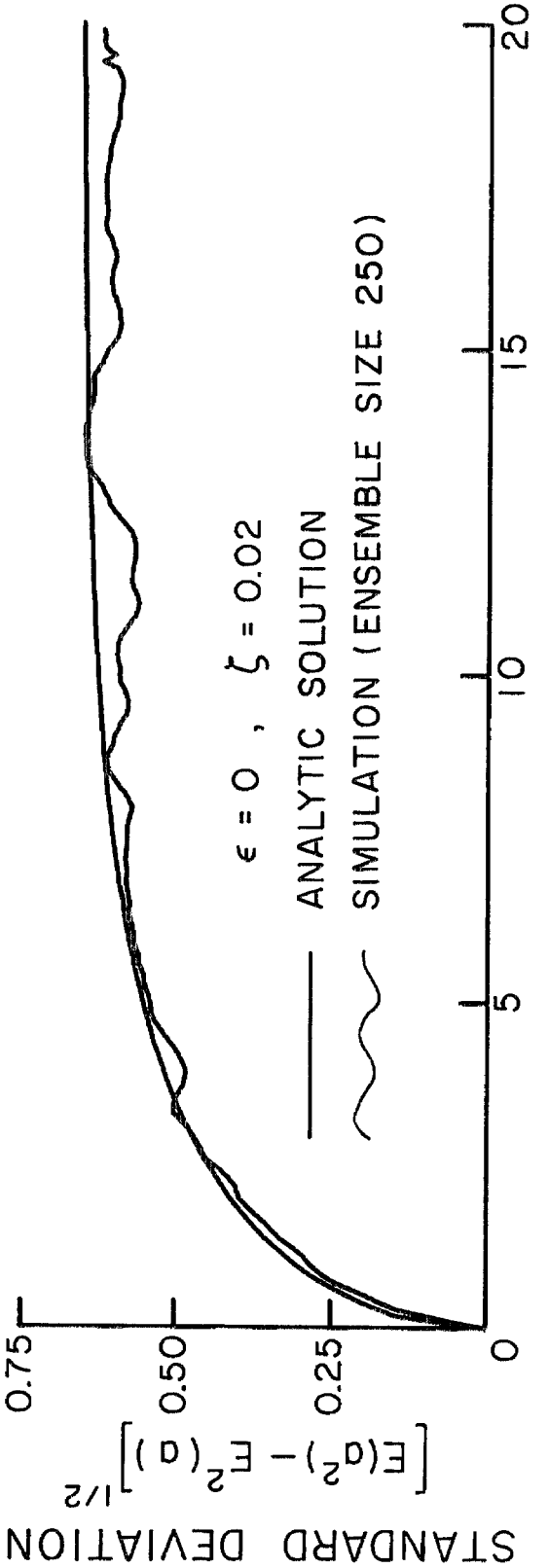


Figure 5.2. Standard Deviation of Non-Stationary Response Amplitude versus Time. Duffing Oscillator, $\epsilon = 0.1$ and $\zeta = 0.02$.

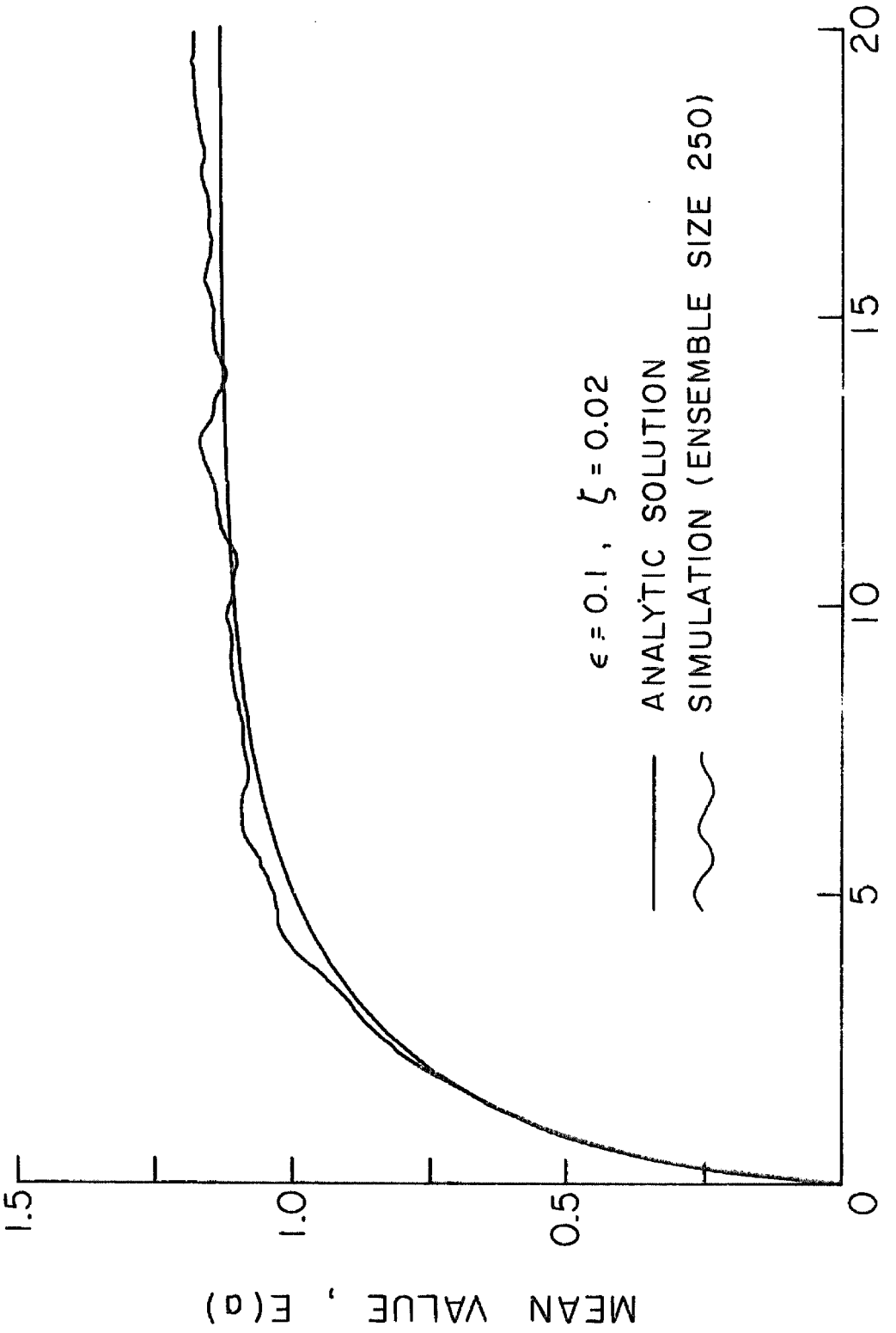


Figure 5.3. Mean Value of Non-Stationary Response Amplitude versus Time. Duffing Oscillator, $\epsilon = 0.1$ and $\zeta = 0.02$.

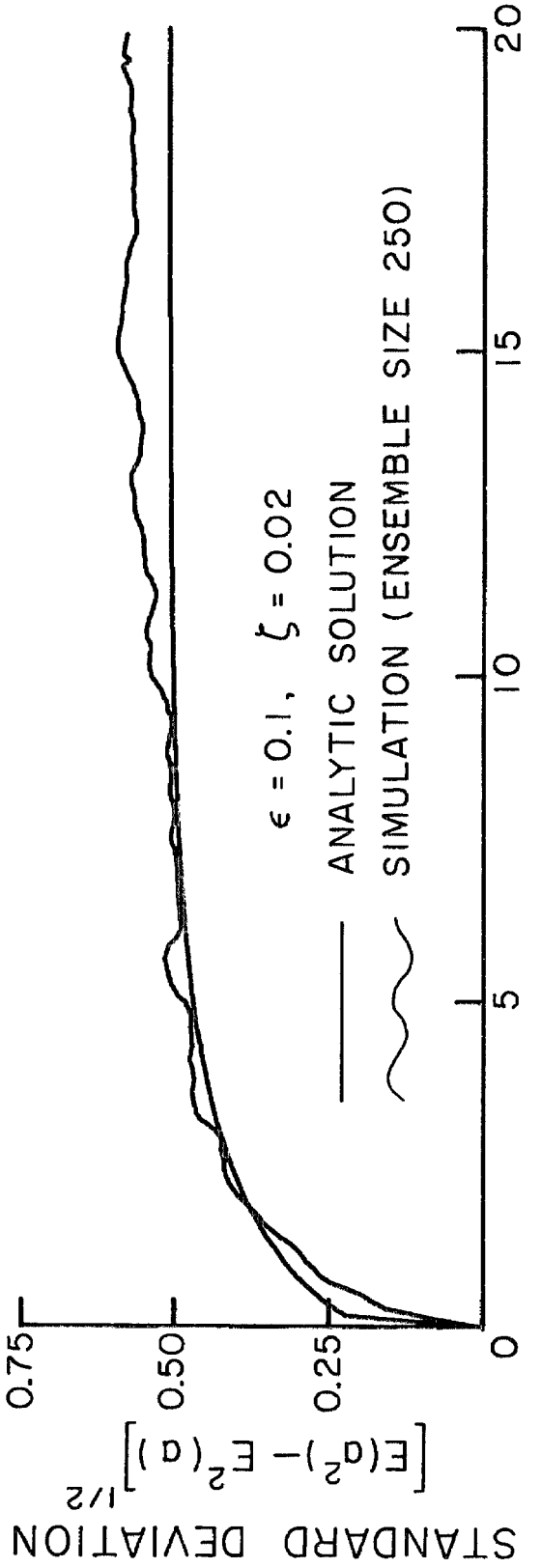


Figure 5.4. Standard Deviation of Non-Stationary Response Amplitude versus Time. Duffing Oscillator, $\epsilon = 0.1$ and $\zeta = 0.02$.

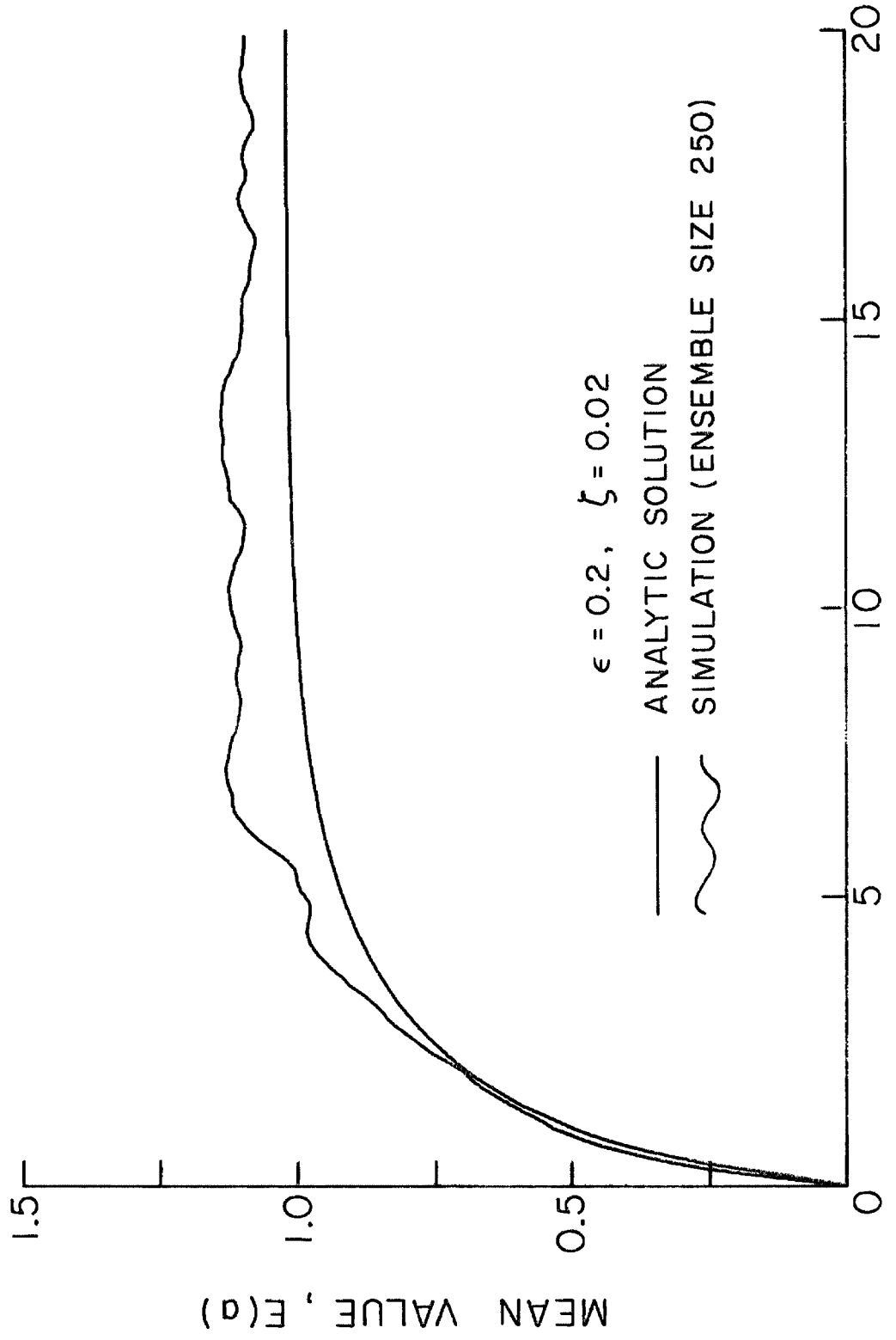


Figure 5.5. Mean Value of Non-Stationary Response Amplitude versus Time, Duffing Oscillator, $\epsilon = 0.2$ and $\zeta = 0.02$.

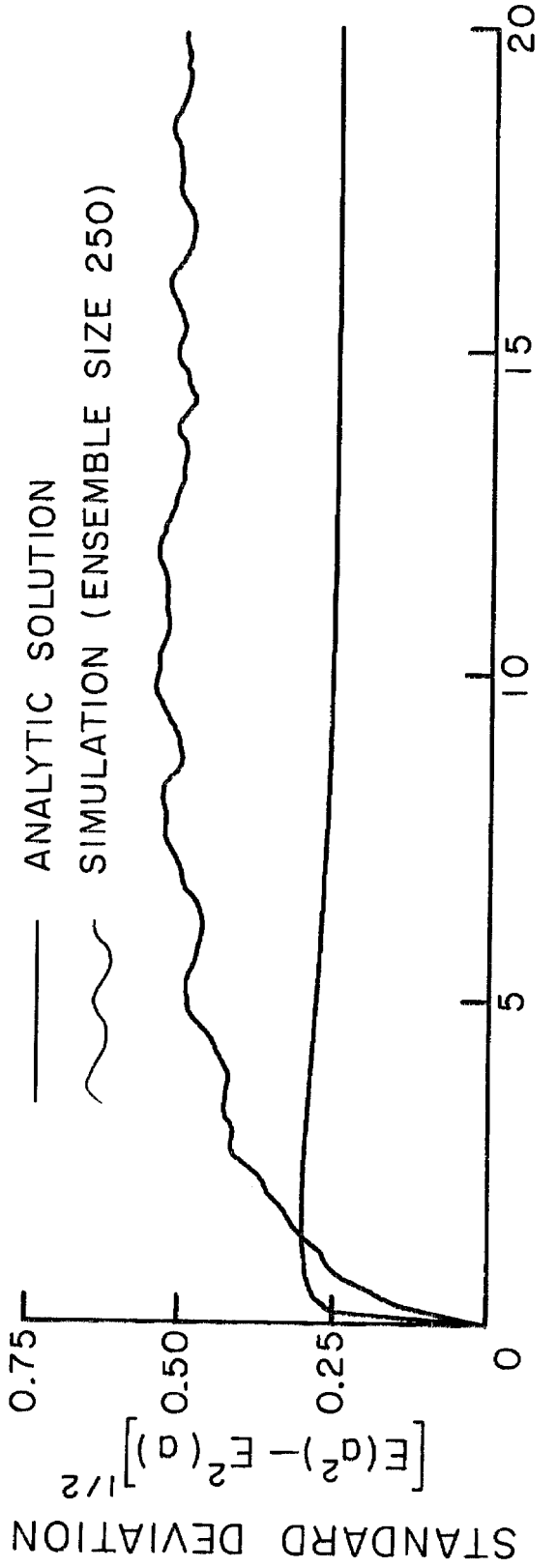


Figure 5.6. Standard Deviation of Non-Stationary Response Amplitude versus Time. Duffing Oscillator, $\epsilon = 0.2$ and $\zeta = 0.02$.

standard deviation for various values of the non-linearity parameter ϵ.

The stationary probability density function of the amplitude $p_s(a)$ is given by equation (5.27). This expression was used for the computation of the stationary mean value and standard deviation of a. The range of ϵ was extended up to values representing severe non-linearities. Figures 5.7 and 5.8 show respectively the stationary mean value and standard deviation of a as functions of ϵ.

For the purpose of checking the results of the present analysis, a simulation study of system (5.3) was performed. The model given in reference [5] was used for simulation of the white Gaussian process. Specifically, for the generation of a sample function of the excitation, a sequence of normally distributed numbers $G_1, \ldots, G_n$ is first generated. Subsequently, the values $G_1, \ldots, G_n$ are assigned to n successive ordinates spaced at equal intervals Δt along a time abscissa. Linear variation of ordinates over each interval is assumed. The initial ordinate G_0 is assumed equal to zero and is located at $t = t_0$, where t_0 is a random variable having a uniform probability density distribution of intensity $1/\Delta t$ over the interval $1 < t_0 < \Delta t$.

A complete ensemble of m such sample functions $a_r(t)$ $(r = 1, \ldots, m)$ can be obtained by repeating the above procedure m times, thereby generating a stationary Gaussian process. If the intensity of this process is changed by multiplying each ordinate G_i by the normalization constant $(2\pi S/\Delta t)^{\frac{1}{2}}$, the power spectral of the

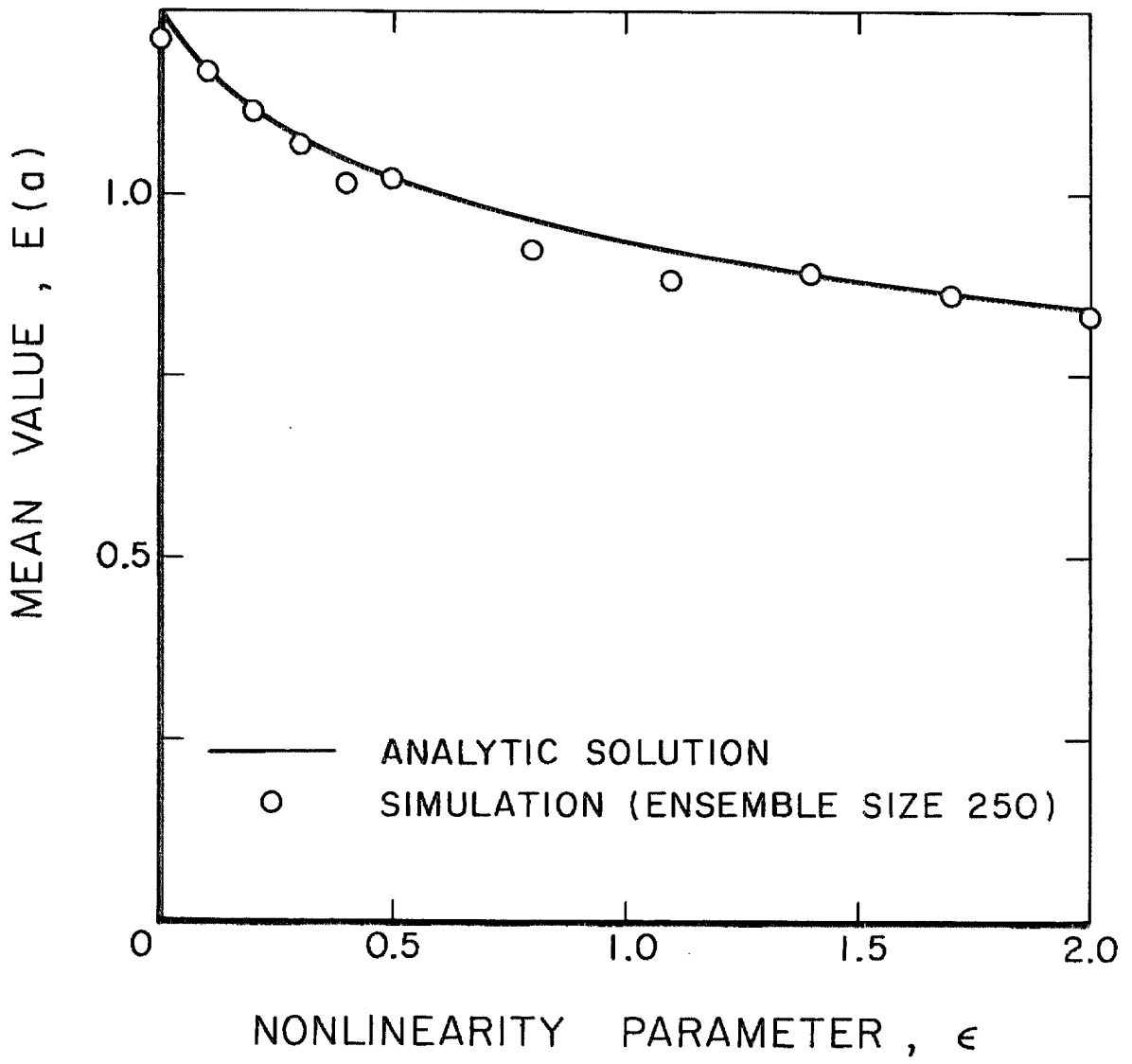


Figure 5.7. Mean Value of Stationary Response Amplitude versus Non-Linearity Parameter. Duffing Oscillator, $\zeta = 0.02$.

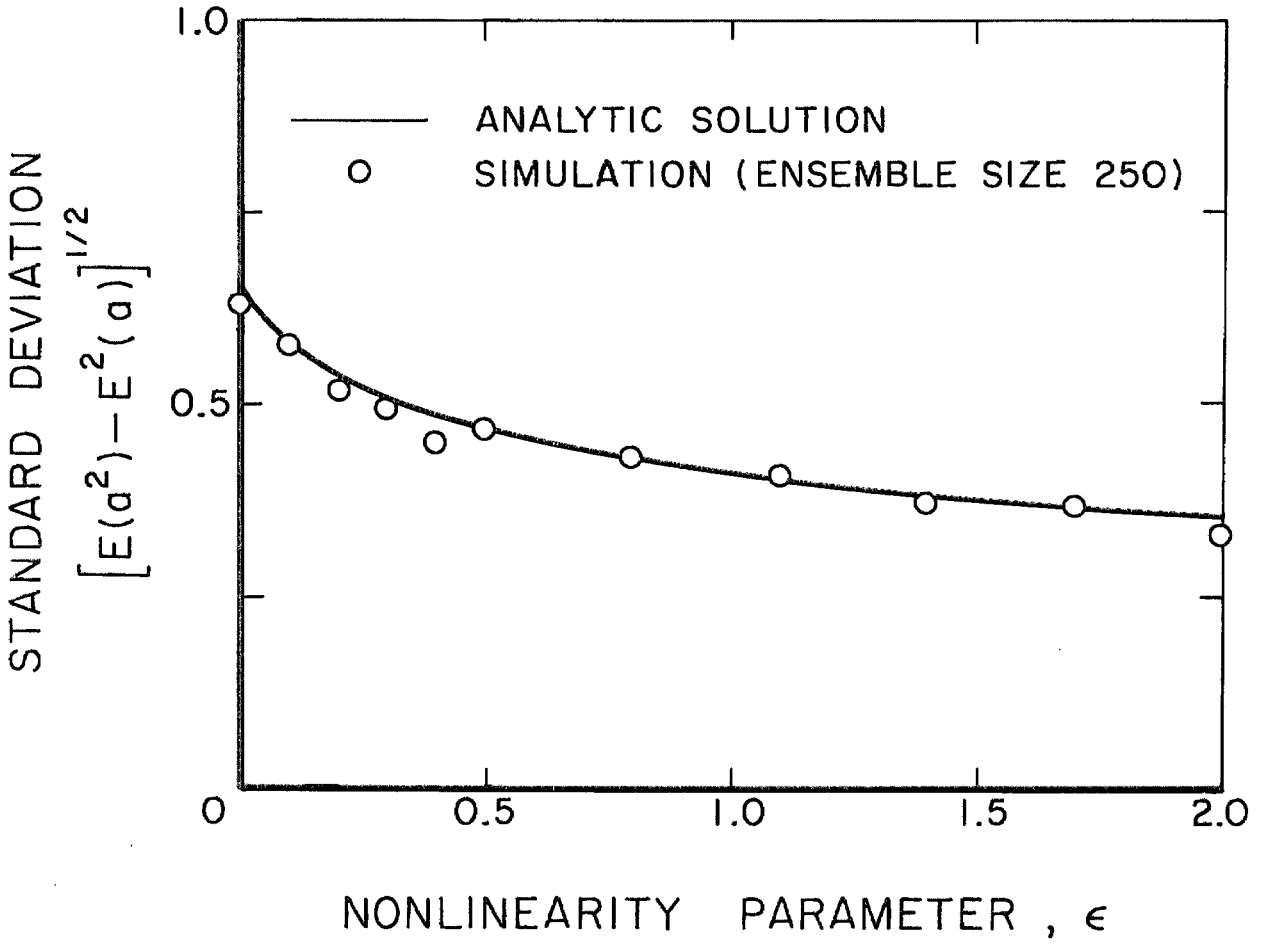


Figure 5.8. Standard Deviation of Stationary Response Amplitude versus Non-Linearity. Duffing Oscillator, $\zeta = 0.02$.

new process is [5]

$$S(\omega) = S \frac{6 - 8 \cos(\omega\Delta t) + 2 \cos(2\omega\Delta t)}{(\omega\Delta t)^4} \qquad -\infty < \omega < \infty \quad . \tag{5.43a}$$

According to reference [5] this function is flat within 5 percent error for $|\omega\Delta t| < 0.57$ and to within 10 percent error for $|\omega\Delta t| < 0.76$. The function drops to 50 percent of its maximum value S at $\omega\Delta t = 2.0$.

For the purpose of the herein discussed simulation study, the dimensionless variable defined by equation (5.41) was introduced in system (5.1). The time interval was taken to be

$$\Delta t' = 0.01 \quad . \tag{5.44}$$

From the simulation study the time dependent mean value and the standard deviation of the amplitude of the response of the non-linear system (5.1) were computed. The same values of the non-linearity parameter ϵ used for the theoretical solution were used for the simulation study as well. The stationary values of the above statistical parameters were also computed for large values of the non-linearity parameter ϵ.

The statistical parameters of the response determined by the simulation procedure are compared with the theoretical solution in Figures 5.1 to 5.8.

A note about the computer time consumed for the computation of the solutions plotted in Figures 5.1 to 5.8 is in order herein. The computation of any of the theoretical non-stationary solutions

required computer time of the order of 10 sec in an IBM370/158 digital computer. For the same problem and by using the same computing machine the generation of an ensemble solution of size 250 required computer time of the order of 1 hour. The numerical computation of the steady-state values of the mean value and the standard deviation by using the solution given by equation (5.17) required computer time of the order of 10 sec.

5.1.5 Discussion

The fact that the non-stationary probability density function of the amplitude of the response given by equation (4.73) is only a first order solution of equation (5.14) is emphasized by Figures 5.1 to 5.6. Specifically, it is seen that as far as the mean value and the standard deviation of the amplitude of the response are concerned, good agreement exists between the theoretical and the simulated response for $\varepsilon = 0.0$ and $\varepsilon = 0.1$ (Figures 5.1 to 5.4). Not only are the proper trends observed but the actual numerical values given by the two approaches are in close agreement. For $\varepsilon = 0.2$ (Figures 5.5 and 5.6) the theoretical solution predicts the correct qualitative nature of the response but the quantitative agreement with the simulation results is rather poor.

It appears therefore that the theoretical solution can be used to obtain reliable quantitative results only for values of ε less than 0.2. This is not a great limitation on the applicability of the method. As far as the stationary values of the mean value and the standard deviation of the amplitude of the response of the

system (5.1) are concerned, the probability density function given by equation (5.27) is more reliable than the solution represented by equation (4.73), especially for large values of the non-linearity parameter ϵ. This is due to the fact that the probability density function given by equation (5.27) does not involve any of the approximations resulting from the perturbation solution of the Fokker-Planck equation.

The stationary values of the mean value and the standard deviation of the normalized amplitude a of system (5.1) are plotted respectively in Figures 5.7 and 5.8, versus the non-linearity parameter ϵ. From Figures 5.7 and 5.8 it is observed that even for large values of ϵ the values of the mean value and the standard deviation determined by the theoretical solution are in good agreement with the corresponding values determined by the simulation study. It is important to note that according to the data represented by Figures 5.7 and 5.8 the theoretical values are conservative in the sense that they are bigger than the corresponding values determined by the simulation study.

5.2 Example of Hysteretic System

5.2.1 General Remarks

The objective of this example-study is to indicate the application of the general method developed in section 4.2 to the non-stationary random response of a hysteretic system.

Structural systems subjected to dynamic loading usually exhibit hysteretic behavior for response amplitudes corresponding to damage.

In order to study the effect of hysteresis on system performance it is desirable to have a mathematically tractable model of the system in question. It must be emphasized that since the restoring force of a hysteretic system depends not only on the instantaneous displacement, but also on its past history, the analytical modeling of such a system under random excitation is not a straightforward matter.

Among the various models available in the technical literature to descirbe the hysteretic behavior, the distributed-element model [39] will be used herein. A lightly damped hysteretic system excited by a Gaussian white noise will be considered. Approximate expressions for the statistics of the non-stationary amplitude of response of the system will be given. All the symbols used herein represent the same quantities as for the general analysis of Chapter IV.

5.2.2 Formulation

Consider a viscously damped hysteretic system excited by a Gaussian white process. The differential equation governing the response of the system is

$$m\ddot{\Psi} + c\dot{\Psi} + h(\Psi, \dot{\Psi}) = w(t) \quad . \tag{5.45}$$

where $h(\Psi, \dot{\Psi})$ represents the hysteretic restoring force. Using transformation (5.2) in equation (5.45) yields

$$m\ddot{x} + c\dot{x} + \frac{h(\Psi, \dot{\Psi})}{\sigma} = \frac{w(t)}{\sigma} \quad . \tag{5.46}$$

It is assumed that the solution x and its derivative dx/dt are given in terms of the amplitude a and the phase θ by equations (4.27) and (4.28).

Before any further application of the general analysis presented in section 4.3 to the non-linear system (5.46), an equivalent stiffness and damping of the hysteretic element $h(\Psi,\dot{\Psi})$ must be determined. For this purpose the distributed element model discussed in reference [39] will be used.

For the development of the distributed element model for the hysteretic system it is assumed that the system is composed of a series of Jenkin's elements. Each Jenkin's element consists of a linear spring with stiffness k/N in series with a Coulomb damper which has a maximum allowable force f_i^*/N, where N is the total number of elements. Clearly, if the loading of the hysteretic system is such that the force on each element does not exceed the corresponding maximum allowable force f_i^*/N, the system behaves linearly with stiffness k. If the load on any of the Jenkin's elements exceeds the corresponding maximum value f_i^*/N the whole system behaves non-linearly.

In reference [39] it is shown that if the total number of elements N becomes very large, it is no longer necessary to know the value of the yield force for each particular member but only the relative occurrence or distribution of the individual yield forces. In this case the hysteretic system will have a total "yield force" given by the expression

$$f_y = \int_0^\infty f^* \varphi(f^*) df^* \qquad (5.47)$$

where $\varphi(f^*)df^*$ is the fraction of the total number of elements having

$f^* \le f_i \le f^* + df^*$. Typical force-deflection loops of a hysteretic system are shown in Figure 5.9.

In reference [39] it is claimed that almost any continuous function $\varphi(f^*)$ could be used for the herein discussed model, provided that the quantity f_y defined by equation (5.47) is finite. For the purpose of the present analysis it will be assumed that the distribution function $\varphi(f^*)$ is of the form shown in Figure 5.10. The distribution function having been selected, the equivalent linear stiffness and damping of the hysteretic system can be determined by direct application of the appropriate formulae given in reference [39]. Specifically, the equivalent damping $c_e^*(a)$ and the equivalent stiffness $k_e^*(a)$ may be calculated by the equations

$$c_e^*(a) = -\frac{k}{\omega_n}\left[\frac{1}{2n}\int_0^{k\sigma a}(2\theta^* - \sin 2\theta^*)\varphi(f^*)df^* + \int_{k\sigma a}^{\infty}\varphi(f^*)df^*\right] \tag{5.48}$$

and

$$k_e^*(a) = -\frac{k}{\pi}\int_0^{k\sigma a}\sin^2\theta^*\varphi(f^*)df^* \ , \tag{5.49}$$

where

$$\theta^* = \cos^{-1}[1 - 2f^*/(kA)] \ . \tag{5.50}$$

Substituting the distribution function $\varphi(f^*)$ shown in Figure 5.10 in equations (5.49) and (5.50) and after some tedious algebraic manipulations, it is found

$$c_e^*(a) = \frac{k}{6\pi\omega_n}\left(\frac{k\sigma}{f_y^*}\right)^2 a^2 \tag{5.51}$$

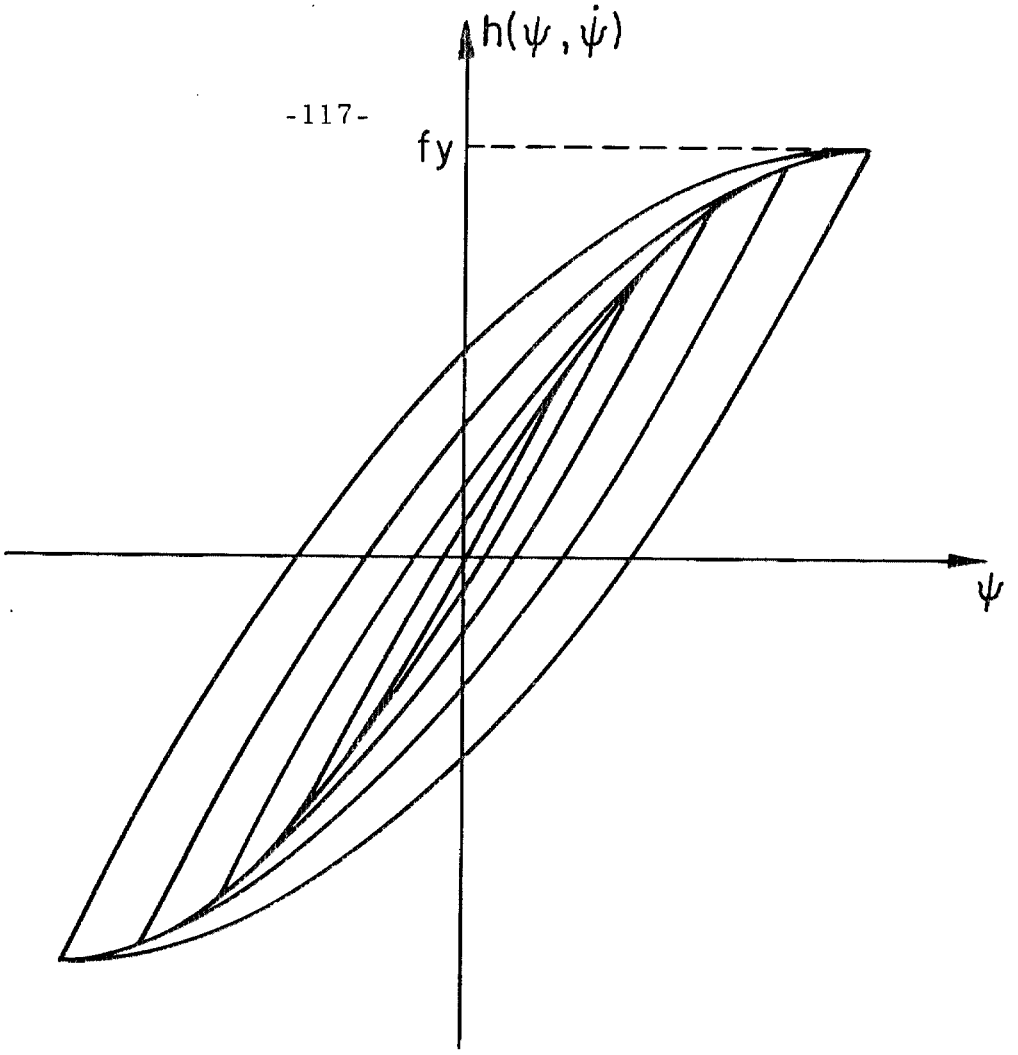


Figure 5.9. Yielding Hysteretic System

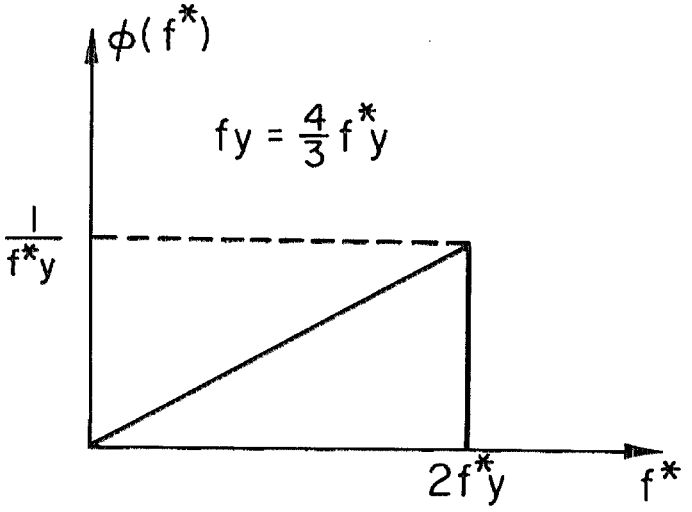


Figure 5.10. Distribution Function for a Hysteretic Model

and

$$k_e^*(a) = k\left[1 - \frac{5}{64}\left(\frac{k\sigma}{f_y^*}\right)^2 a^2\right] \quad . \tag{5.52}$$

It is also found that

$$f_y = \frac{4}{3} f_y^* \quad . \tag{5.53}$$

The equivalent linear system for the hysteretic system (5.46) is

$$m\ddot{x} + (c + c_e^*)\dot{x} + k_e^* x = \frac{w(t)}{\sigma} \quad . \tag{5.54}$$

It will be assumed that the spectral density S of the excitation and the quantities k and f_y of the hysteretic system are such that

$$k\sigma/f_y^* \ll 1 \quad . \tag{5.55}$$

Physically, assumption (5.55) implies that the system is far from the failure state. Define the non-linearity parameter ϵ and the symbols c_e and k_e as

$$\epsilon = \left(\frac{k\sigma}{f_y^*}\right)^2 \tag{5.56}$$

$$c_e = \frac{k}{6\pi\omega_n} a^2 \tag{5.57}$$

and

$$k_e = -\frac{5k}{64} a^2 \quad . \tag{5.58}$$

Using the definitions expressed by equations (5.56), (5.57) and (5.58) the system (5.46) may be rewritten as

$$m\ddot{x} + (c + \epsilon c_e)\dot{x} + (k + \epsilon k_e)x = \frac{w(t)}{\sigma} \quad . \tag{5.59}$$

The natural angular frequency of the equivalent linear system (5.54) is

$$\omega_n^2 = \omega_0^2 \left(1 - \frac{5}{64}\epsilon a^2\right) \quad . \tag{5.60}$$

The stochastic differential equation governing the amplitude of the response of system (5.59) is given by the general formula (4.37). This gives

$$\dot{a} = -\frac{c}{2m} a - \epsilon \frac{k/6\pi\omega_n}{2m} a^3 + \frac{\pi S}{2am^2\omega_n^2\sigma^2} + \frac{(\pi S)^{\frac{1}{2}}}{\omega_n m\sigma} \eta(t) \tag{5.61}$$

where $\eta(t)$ is a delta correlated random process with zero mean. Introducing the ratio of critical damping ζ and substituting equation (5.60) in equation (5.61) yields

$$\dot{a} = -\zeta\omega_0\left(a - \frac{1}{a\left(1 - \frac{5}{64}\epsilon a^2\right)}\right) - \epsilon \frac{c_e(a)}{2m} a + \left(\frac{2\zeta\omega_0}{1 - \frac{5}{64}\epsilon a^2}\right)^{\frac{1}{2}} \eta(t) \quad . \tag{5.62}$$

5.2.3 Fokker-Planck Equation

The Fokker-Planck equation associated with equation (5.62) is

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left\{p\left[\zeta\omega_0\left(a - \frac{1}{a\left(1 - \frac{5}{64}\epsilon a^2\right)}\right) + \epsilon \frac{c_e(a)}{2m} a\right]\right\}$$

$$+ \frac{\partial}{\partial a}\left\{\frac{\zeta\omega_0}{1 - \frac{5}{64}\epsilon a^2} \frac{\partial p}{\partial a} + \tfrac{1}{2}\zeta\omega_o p \frac{\partial}{\partial a}\left(\frac{1}{1 - \frac{5}{64}\epsilon a^2}\right)\right\} \quad . \tag{5.63}$$

A first order non-stationary solution of equation (5.63) will be determined. This solution will be valid for those values of a such that

$$\frac{5}{64}\epsilon a^2 \ll 1 \quad . \tag{5.64}$$

Because of the assumption expressed by equation (5.64), the following first order approximations can be made

$$\epsilon \frac{c_e(a)}{2m} = \epsilon \frac{\omega_0}{12\pi} \tag{5.65}$$

and

$$\frac{1}{1 - \frac{5}{64}\epsilon a^2} = 1 + \frac{5}{64}\epsilon a^2 \quad . \tag{5.66}$$

By virtue of approximations expressed by equations (5.65) and (5.66) the Fokker-Planck equation (5.63) can be written as

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \zeta\omega_0\left[\frac{\partial^2 p}{\partial a^2} + \frac{\partial}{\partial a}(ap) - \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right] \\
&\quad - \frac{5}{64}\zeta\omega_0\epsilon\left[\frac{\partial(ap)}{\partial a} - \frac{\partial}{\partial a}\left(a^2\frac{\partial p}{\partial a}\right) - \frac{\partial(ap)}{\partial a}\right] + \epsilon\frac{\omega_0}{12\pi}\frac{\partial}{\partial a}(a^3 p) \\
&= \zeta\omega_0\left[\frac{\partial^2 p}{\partial a^2} + \frac{\partial}{\partial a}(ap) - \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right] + \frac{5}{64}\zeta\omega_0\epsilon\frac{\partial}{\partial a}\left(a^2\frac{dp}{\partial a}\right) \\
&\quad + \epsilon\frac{\omega_0}{12\pi}\frac{\partial}{\partial a}(a^2 p) \quad .
\end{aligned} \tag{5.67}$$

Herein, it is assumed that the boundary and initial conditions expressed by equations (4.46) and (4.56) hold for the function p(a,t). For the purpose of determining a first order solution of equation (5.67)

the coefficients ρ_m and $\beta_{m,k}$ given by formulae (4.72) and (4.72a) must be calculated. To achieve this, the form of the general expression H_m introduced in section (4.2) by equation (4.66) must be determined for the present problem. Considering equation (5.66) it is readily verified that

$$H_m = -\frac{5}{64}\zeta\omega_0 \frac{d}{da}\left(a^2 \frac{dA_m}{da}\right) - \frac{\omega_0}{12\pi}\frac{d}{da}(a^3 A_m) \; ; \quad m = 0,\ldots \; . \tag{5.68}$$

Comparing equation (5.68) with equation (5.16) it is noticed that the first term of the sum on the right hand side of equation (5.68) can be obtained by multiplying the right hand side of equation (5.16) by -5/48. Therefore, the contributions ρ^*_m and $\beta^*_{m,k}$ of the term $-\frac{5}{64}\zeta\omega_0 \frac{d}{da}(a^2 A_m)$ to the coefficients ρ_m and $\beta_{m,k}$ can be readily determined by multiplying the right hand sides of equations (5.19), (5.20), (5.21), (5.22) and (5.23) by -5/48. In this manner it is found that

$$\rho^*_m = -\frac{5}{32}\zeta\omega_0 m(2m+1) \; ; \qquad m = 0,\ldots \tag{5.69}$$

$$\beta^*_{m,m+1} = -\frac{5}{64}(m+1)(4m+3) \; ; \qquad m = 0,\ldots \tag{5.70}$$

$$\beta^*_{m,m+2} = \frac{5}{64}(m+1)(m+2) \; ; \qquad m = 0,\ldots \tag{5.71}$$

$$\beta^*_{m,k} = 0 \; ; \qquad k \neq m+2, m+1, m \tag{5.72}$$

$$\beta^*_{m,m} = -\frac{15}{64} m^2 \; ; \qquad m = 0,\ldots \; . \tag{5.73}$$

The second term of the sum of the right hand side of equation (5.68) is

$$\overline{H}_m = -\frac{\omega_0}{12\pi}\frac{d}{da}(a^3 A_m) \quad . \tag{5.74}$$

Using formula (B25) for the derivative dA_n/da, equation (5.74) can further be manipulated to yield

$$\overline{H}_m = -\frac{\omega_0}{12\pi}\left(a^3\frac{dA_n}{da} + 3a^2 A_n\right) = -\frac{\omega_0}{6\pi}a^2[(n+1)A_{n+1} - (n-1)A_n] \quad . \tag{5.75}$$

Next the coefficients $\overline{\rho}_m$ and $\overline{\beta}_{m,k}$ associated with the function $\overline{H}_m$ will be determined. Using the general formula (4.72) and the relations (B13) and (B14) it is found that

$$\overline{\rho}_m = \int_0^\infty \frac{H_m A_m}{A_0}\,da = \frac{\omega_0}{3\pi}\,m(m+1) \quad ; \qquad m = 0, \ldots \quad . \tag{5.76}$$

Applying the general formula (4.72a) to the present problem and using relations (B13), B(14) and (B15) yields

$$\overline{\beta}_{m,m+2} = \frac{1}{\lambda_m - \lambda_{m+2}}\int_0^\infty \frac{\overline{H}_m A_{m+1}}{A_0}\,da = -\frac{1}{12\pi\zeta}(m+1)(m+2) \tag{5.77}$$

$$\overline{\beta}_{m,m+1} = \frac{1}{\lambda_m - \lambda_{m+1}}\int_0^\infty \frac{\overline{H}_m A_{m+2}}{A_0}\,da$$

$$= \frac{1}{6\pi\zeta}(m+1)(3m+2) \quad ; \qquad m = 0, \ldots \tag{5.78}$$

$$\overline{\beta}_{m,m-1} = \frac{1}{\lambda_m - \lambda_{m-1}}\int_0^\infty \frac{\overline{H}_m A_{m-1}}{A_0}\,da = -\frac{1}{6\pi\zeta}m(m-1) \tag{5.79}$$

$$\bar{\beta}_{m,k} = 0 \quad ; \qquad k \neq m+2, m+1, m, m-1 \quad . \tag{5.80}$$

Using herein the general formulae (4.80) yields

$$\bar{\beta}_{m,m} = -(\beta_{m-2,m} + \beta_{m-1,m} + \beta_{m+1,m}) = -\frac{1}{4\pi\zeta} m(m-1) \quad ;$$

$$m = 0, \ldots \quad . \tag{5.81}$$

The coefficients ρ^*_m, $\bar{\rho}_m$, $\beta^*_{m,k}$, and $\bar{\beta}_{m,k}$ being calculated, the first order solution of equation (5.64) is given by equation (4.73), where

$$\rho_m = \rho^*_m + \bar{\rho}_m \quad ; \qquad m = 0, \ldots \tag{5.82}$$

and

$$\beta_{m,k} = \beta^*_{m,k} + \bar{\beta}_{m,k} \quad ; \qquad m = 0, \ldots \quad . \tag{5.83}$$

5.2.4 Discussion

Expression (4.73) was used for the numerical computation of the normalized by σ non-stationary mean value E(a) of the amplitude of the response of the non-linear hysteretic system (5.45). The discussion presented in part (5.1.4) concerning the actual implementation of the numerical calculations related to the Duffing oscillator, covers the case of the hysteretic system (5.45) as well.

The results of the numerical computations are shown in Figure 5.11 for various values of the non-linearity parameter $k\sigma/f^*_y$. The value of the ratio of critical viscous damping was taken to be

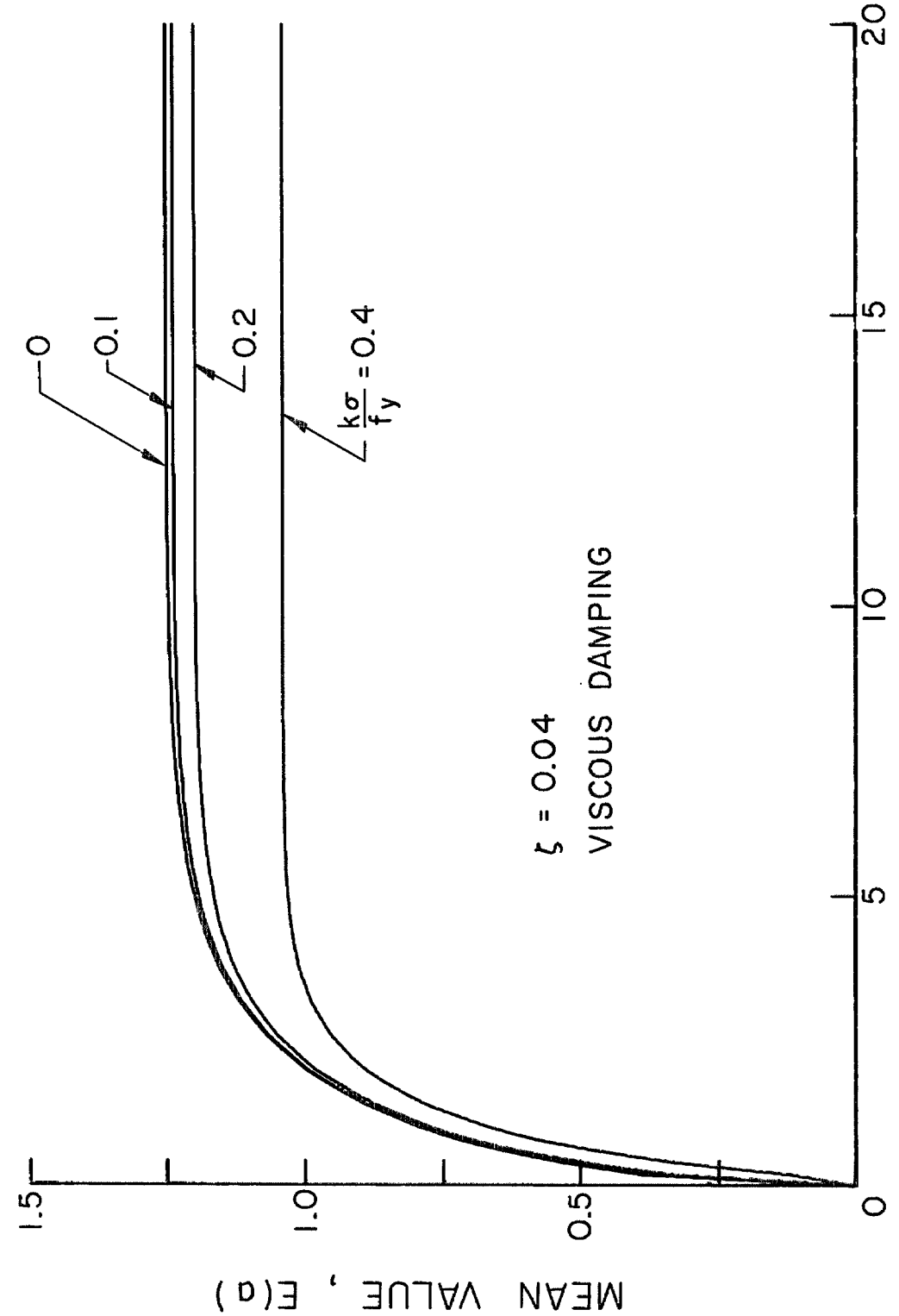


Figure 5.11. Mean Value of Non-Stationary Response Amplitude versus Time. Hysteretic System, $\zeta = 0.04$.

$$\zeta = 0.04 \ . \tag{5.83}$$

From Figure 5.11 it is seen that the higher the non-linearity parameter, the smaller the mean-value of the amplitude of the response of the hysteretic system, and the faster a steady-state value is reached. These observations may be interpreted in the context of the physical meaning of the equivalent linear stiffness and viscous damping of the model for the hysteretic system.

From equation (5.52) it is seen that the hysteretic system represents a softening type non-linearity. This means that the stiffness of the system becomes weaker as the non-linear parameter $k\sigma/f_y^*$ is increased. This characteristic of the model, considered independently of any other effects, would imply an increase in the amplitude of the response as the parameter $k\sigma/f_y$ is increased. It can easily be verified from equation (5.51) that the equivalent viscous damping also increases as $k\sigma/f_y$ increases. This effect tends to cause a decrease in the response amplitude with increasing value of the nonlinearity parameter. Therefore, the softening of the stiffness is counteracted by an increase in the energy dissipation as the value of the non-linearity parameter increases.

The softening-spring effect and the energy dissipation due to yielding have been discussed in references [33] and [35] in the context of the hysteretic response of structures to earthquakes. The same effect has been discussed in reference [36] in the context of the stationary response of a bilinear hysteretic system. In

that study, it has been found by electronic-analog techniques that for large values of the non-linearity parameter $k\sigma/f_y$ the softening-spring effect dominates and the overall response is increased. This trend of the response of a hysteretic system as the non-linearity effect becomes large cannot be examined by the model presented herein. This is due to the fact that formulae (5.51) and (5.52) providing the equivalent linear stiffness and viscous damping of the hysteretic system are valid, according to reference [39], only for small values of the non-linearity parameter $k\sigma/f_y$.

For the range of values of the parameter $k\sigma/f_y$ considered in the present study, data on the stationary response of a bilinear hysteretic system [36] indicate that the energy dissipation due to yielding dominates the spring-softening effect. Hence, the overall response of the system decreases with increasing $k\sigma/f_y$. This experimental evidence supports the trend predicted by the theoretical results presented herein.

The domination of the energy dissipation due to yielding, for small values of the non-linearity parameter, may also be a plausible explanation of the observed faster rise time as the parameter $k\sigma/f_y$ is increased. This effect of the non-linearity parameter on the rise time of a hysteretic system has also been noticed during simulation studies of the non-stationary response of a bilinear hysteretic system excited by a Gaussian white process [37].

According to the preceding analysis the trend of the theoretical solution shown in Figure 5.11 agrees with the available

experimental results in that it predicts the same general behavior of the hysteretic system for small values of the non-linearity parameter $k\sigma/f_y$.

It appears that there are no theoretical or experimental investigations which can be compared directly with the results of the present study. This is due to the fact that to date very limited research has been performed towards the determination of the amplitude of the non-stationary response of a non-linear system subjected to random excitation. In reference [38] Monte Carlo simulation was used to generate the statistics of the maximum of the non-stationary response of a bilinear hysteretic system excited by a Gaussian white process. Unfortunately, besides the fact that a different model for the hysteretic system was used, rather large values were assigned to the non-linearity parameter of the problem. Consequently, there is no meaningful way to compare quantitatively the results of the simulation study with the results of the present study which were derived by assuming small values for the non-linearity parameter $k\sigma/f_y$.

VI. Concluding Remarks

Due to the fact that rather detailed summaries have been given at the end of each of the main sections of the preceding chapters, the goal of this chapter will be a critical evaluation of the potential of the generalized equivalent linearization method. Only a general synopsis of the work presented in the preceding pages will be included herein.

In Chapter II a general method for determining an approximate solution of a multi-degree-of-freedom non-linear dynamical system is discussed. The principle of the method is the approximation of the exact solution of the non-linear system by the solution of a mathematically tractable equivalent linear system which is as close to the original as possible. The last requirement is satisfied in Chapter II by minimizing an average of the Euclidean norm of the difference between the linear and the equivalent linear system. It is understood that different criteria will result in different equivalent linear systems. Consequently, the determination of the best criterion must be subject to further investigation.

From the analysis performed in Chapter II it is seen that the elements of the equivalent linear system depend on the identification parameters of the members of the class of approximate solutions. In that sense, it is inherently assumed in the formulation of the method, that the approximate solution belongs to a class of functions or processes each member of which may be identified by a relatively small number of parameters. This is true for example for the classes of harmonic functions, Gaussian and

narrow-band random processes. However, this is not the general rule. For example, to date it has not been possible to approximate properly the transient solution of a multi-degree-of-freedom non-linear system by a finite number of parameters. Therefore, it appears that investigations directed toward broadening the class of functions identifiable by a small number of parameters is in order.

The problem of obtaining a theoretical estimation of the accuracy of the method is very formidable. For a specific problem it appears more realistic to compare the approximate solutions determined by the method with an "exact" solution computed numerically.

In Chapter III the method is considered in the context of the stationary response of a multi-degree-of-freedom system excited by a harmonic monofrequency or by a stationary Gaussian random vector. The identification parameters of the approximate solution of the harmonically excited system are the amplitudes and the phases of each component of the solution. Ultimately, non-linear algebraic equations for the determination of the amplitudes and the phases of the response are generated by the method of equivalent linearization. Clearly, if all of the components of the forcing function were not oscillating at the same frequency, the components of the steady-state response would no longer exhibit monofrequency harmonic behavior described by an amplitude and a phase. This note emphasizes the dependence of the applicability of the method on the form of the approximation solutions which can be postulated for a specific class of problems.

The elements of the covariance matrix are taken as the identification parameters of the solution of the multi-degree-of-freedom non-linear system subjected to Gaussian random excitation. Ultimately, non-linear algebraic equations are derived for the determination of these elements by applying the generalized method of equivalent linearization.

It is interesting to note that the method of equivalent linearization cannot be applied to multi-degree-of-freedom hysteretic systems excited by a Gaussian excitation. This is due to the fact that to date it has not been possible to postulate an approximate solution form which is on the one hand mathematically tractable and on the other hand accounts for the complexities of the hysteretic system.

Study of the harmonic and the Gaussian response of a multi-degree-of-freedom non-linear system has indicated that the existence and the uniqueness of an equivalent linear system depend on the properties of the class of approximate solutions. In addition, in Chapter III it was shown that the properties of the class of approximate solutions can occasionally be used for a more direct determination of the equivalent linear system.

In Chapter IV the present method is applied to the study of the transient response of a lightly damped and weakly non-linear oscillator subjected to a harmonic monofrequency or to a Gaussian white process. The key to the development is the assumption that the response is pseudo-sinusoidal with slowly varying amplitude and phase. This assumption allows the construction of an

equivalent linear system which in turn may be used to derive a first-order differential equation describing the response amplitude as a function of time. It is thought that the interpretation of this equation in terms of the energy of the equivalent linear oscillator contributes to a better understanding of the engineering aspects of the problem. The results of the present investigation indicate that no conceptual difficulties should be anticipated in any future extension of the energy method to problems involving non-white and non-stationary excitations.

The examples studied in Chapter VI serve to indicate the qualitative and quantitative reliability of the method. The qualitative results for both the Duffing oscillator and the Hysteretic System are in agreement with trends that would be expected based on common engineering considerations. In addition, the numerical results for the Duffing oscillator are in agreement with corresponding results generated by a Monte Carlo simulation study. This simulation study emphasizes the first-order nature of the non-stationary solution, while demonstrating the applicability of the stationary solution to dynamical systems with quite severe non-linearities.

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Appendix A

Simplified Equations for the Amplitude and the Phase of the Response of a Randomly Excited Linear Oscillator

Reference [56] has been used extensively for this appendix.

Consider the equation of motion of a linear oscillator

$$m\ddot{x} + c\dot{x} + kx = w(t) \tag{A1}$$

where w(t) is a stationary wide band random process with

$$E[w(t)] = 0 \tag{A2}$$

$$E[w(t)\,w(t+\tau)] = K(\tau) \tag{A3}$$

and spectral density $S(\omega)$ given by

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\tau) e^{-i\omega t}\, d\tau \ . \tag{A4}$$

The natural angular velocity ω_n and the ratio of critical damping ζ are given by

$$\omega_n^2 = \frac{k}{m} \tag{A5}$$

and

$$2\zeta\omega_n = \frac{c}{m} \ . \tag{A6}$$

The variables a and θ are introduced as

$$x = a \cos(\omega_n t + \theta) \tag{A7}$$

and

$$\dot{x} = -a\omega_n \sin(\omega_n t + \theta) \quad . \tag{A8}$$

It is seen that transformations (A7) and (A8) represent the relations which would hold for a sinusoidal wave and its derivative in terms of the amplitude a and the phase θ. Solving equations (A7) and (A8), it is easily shown that

$$a = \left(x^2 + \frac{\dot{x}^2}{\omega_n^2} \right)^{\frac{1}{2}} \tag{A9}$$

and

$$\theta = \tan^{-1}\left(\frac{\dot{x}}{\omega_n}\right) - \omega_n t \quad . \tag{A10}$$

Differentiating equation (A9) gives

$$\dot{a} = \frac{da}{dt} = \frac{\dot{x}\ddot{x} + \frac{\dot{x}\dddot{x}}{\omega_n^2}}{a} \quad . \tag{A11}$$

Using equation (A1) the last equation can be rewritten as

$$\dot{a} = -\left[\frac{c}{m\omega_n^2}\dot{x} + \frac{w(t)}{m\omega_n^2} \right] \frac{\dot{x}}{a} \quad . \tag{A12}$$

Because of transformation (A8), equation (A12) can be put in the form

$$\dot{a} = -\frac{c}{m} a \sin^2(\omega_n t + \theta) - \frac{w(t)}{m\omega_n} \sin(\omega_n t + \theta) \quad . \tag{A13}$$

Differentiating equation (A14) it is seen that

$$\dot{\theta} = \frac{d\theta}{dt} = - \frac{\frac{\ddot{x}}{\omega_n x} - \frac{\dot{x}^2}{\omega_n x^2}}{1 + \frac{\dot{x}^2}{\omega_n^2 x^2}} - \omega_n \quad . \tag{A14}$$

Using equation (A1), this equation may be written as

$$\dot{\theta} = - \left[\frac{c}{m} \dot{x} + \frac{w(t)}{m} \right] \frac{x}{a^2 \omega_n} \quad . \tag{A15}$$

Because of transformation (A7), equation (A15) can be put in the form

$$\dot{\theta} = - \frac{c}{m\omega_n} \sin (\omega_n t + \theta) \cos (\omega_n t + \theta) - \frac{w(t)}{ma\omega_n} \cos (\omega_n t + \theta) \quad . \tag{A16}$$

At this stage, additional assumptions about the problem are made. It is assumed that the damping of the system and the spectral density of the excitation are small. Mathematically these assumptions may be expressed as

$$\zeta \ll 1 \tag{A17}$$

and

$$S = 0(\zeta) \quad . \tag{A18}$$

It is noticed that the right-hand sides of equations (A13) and (A14) are periodic with respect to the variable θ, hence $a = 0(\zeta)$ and $\dot{\theta} = 0(\zeta)$. Thus, a and θ are slowly varying functions of t because

ζ is small; hence, they change very little during the time $T = 2\pi/\omega_n$. Therefore, the quantities $\sin^2(\omega_n t + \theta)$, $\cos^2(\omega_n t + \theta)$ and $\sin(\omega_n t + \theta)\cos(\omega_n t + \theta)$ may be approximated by their averages over the interval $[t, t+T]$. Doing so, equations (A13) and (A16) can be rewritten as

$$\dot{a} = -\frac{c}{2m} a - \frac{w(t)}{m\omega_n} \sin(\omega_n t + \theta) \tag{A19}$$

and

$$\dot{\theta} = -\frac{w(t)}{ma\omega_n} \cos(\omega_n t + \theta) \quad . \tag{A20}$$

Attention is now focused on the processes

$$\eta_1(t) = -w(t) \sin(\omega_n t + \theta) \tag{A21}$$

and

$$\xi_1(t) = -w(t) \cos(\omega_n t + \theta) \quad . \tag{A22}$$

The analysis given in reference [56] will serve as a guide. More rigorous discussions may be found in references [57] and [29].

Obviously the phase θ is correlated with the values of the random function w(t). This fact complicates in general the calculation of the statistical parameters of $\eta_1(t)$ and $\xi_1(t)$. However, w(t) is a wide band random process; therefore, its autocorrelation function $K(\tau)$ is non-zero for values of τ only very near to zero. In fact if w(t) is white noise, it is delta-correlated. Therefore, the values of a and θ which correspond to slightly shifted times $t \pm \Delta t$ will be effectively statistically independent of w(t), and hence it is possible

to perform an ensemble average over w(t) and $\theta_{-\Delta t}$ separately. The symbol $\theta_{-\Delta t}$ is defined as

$$\theta_{-\Delta t} = \theta(t - \Delta t) = \theta - \Delta\theta = \theta - \omega_n \Delta t \ . \tag{A23}$$

Equations (A21) and (A22) can be written as

$$\begin{aligned} \eta_1(t) &= -w(t) \sin(\omega_n t + \theta_{-\Delta t} + \Delta\theta) \\ &= -w(t)[\sin(\omega_n t + \theta_{-\Delta t}) \cos\Delta\theta + \cos(\omega_n t + \theta_{-\Delta t}) \sin\Delta\theta] \end{aligned} \tag{A24}$$

and

$$\begin{aligned} \xi_1(t) &= -w(t) \cos(\omega_n t + \theta_{-\Delta t} + \Delta\theta) \\ &= -w(t)[\cos(\omega_n t + \theta_{-\Delta t}) \cos\Delta\theta - \sin(\omega_n t + \theta_{-\Delta t}) \sin\Delta\theta] \ . \end{aligned} \tag{A25}$$

Since Δt is assumed to be small

$$\cos\Delta\theta \approx 1 \tag{A26}$$

and

$$\sin\Delta\theta \approx \Delta\theta \ . \tag{A27}$$

Therefore equations (A25) and (A26) can be rewritten respectively as

$$\eta_1(t) \approx -w(t)[\sin(\omega_n t + \theta_{-\Delta t}) + \cos(\omega_n t + \theta_{-\Delta t})\Delta\theta] \tag{A28}$$

and

$$\xi_1(t) \approx -w(t)[\cos(\omega_n t + \theta_{-\Delta t}) - \sin(\omega_n t + \theta_{-\Delta t})\Delta\theta] \ . \tag{A29}$$

Because of relation (A2) and the statistical independence of $w(t)$ and $\theta_{-\Delta t}$, the following expressions for the expectations of the processes η_1 and ξ_1 may be derived

$$E(\eta_1) = -\cos(\omega_n t + \theta_{-\Delta t})\, E(w\Delta\theta) \tag{A30}$$

and

$$E(\xi_1) = -\sin(\omega_n t + \theta_{-\Delta t})\, E(w\Delta\theta) \quad . \tag{A31}$$

Since the amplitude $a(t)$ and the phase $\theta(t)$ are slowly varying processes, it can be assumed that they do not manage to change appreciably during the small time interval Δt. Therefore, it is possible to replace $\theta_{-\Delta t}$ by θ in (A30) and (A31). In that case, equations (A30) and (A31) can be rewritten as

$$E(\eta_1) = -\cos(\omega_n t + \theta)\, E(w\Delta\theta) \tag{A32}$$

and

$$E(\xi_1) = -\sin(\omega_n t + \theta)\, E(w\Delta\theta) \quad . \tag{A33}$$

Next, equation (A20) for the phase is integrated giving

$$2\Delta\theta = -\frac{1}{m\omega_n} \int_{t-\Delta t}^{t+\Delta t} \frac{w(s)}{a} \cos(\omega_n s + \theta)\, ds$$

$$= -\frac{1}{m\omega_n} \int_{-\Delta t}^{\Delta t} \frac{w(t+\tau)}{a} \cos[\omega_n(t+\tau) + \theta]\, d\tau \tag{A34}$$

which implies

$$E(w\Delta\theta) = -\frac{1}{2m\omega_n}\int_{-\Delta t}^{\Delta t} E\left[\frac{w(t)w(t+\tau)}{a}\cos\left[\omega_n(t+\tau)+\theta\right]\right]d\tau \quad . \qquad (A35)$$

Once again exploiting the fact that $a(t)$ and $\theta(t)$ differ only slightly from the values $a_{\pm\Delta t}$ and $\theta_{\pm\Delta t}$ which are effectively independent of $w(t)$ and $w(t+\tau)$, equation (A35) can be rewritten as

$$E(w\Delta\theta) = -\frac{1}{2m\omega_n a}\int_{-\Delta t}^{\Delta t} E[w(t)w(t+\tau)]\cos\left[\omega_n(t+\tau)+\theta\right]d\tau \quad . \qquad (A36)$$

The limits of integration in equation (A36) may be extended to $\pm\infty$ because the process $w(t)$ being wide band, the autocorrelation function $E[w(t)w(t+\tau)]$ is essentially zero for $|\tau| > |\Delta t|$. According to this approximation, equation (A36) can be rewritten as

$$E(w\Delta\theta) = -\frac{1}{2m\omega_n a}\int_{-\infty}^{\infty} E[w(t)w(t+\tau)]\cos\left[\omega_n(t+\tau)+\theta\right]d\tau \quad . \qquad (A37)$$

Using expression (A37) for $E(w\Delta\theta)$ in equations (A32) and (A33) gives

$$E(\eta_1) = \frac{1}{2m\omega_n a}\int_{-\infty}^{\infty} E[w(t)w(t+\tau)]\cos\left[\omega_n(t+\tau)+\theta\right]\cos(\omega_n t+\theta)d\tau \quad . \qquad (A38)$$

and

$$E(\xi_1) = \frac{1}{2m\omega_n a}\int_{-\infty}^{\infty} E[w(t)w(t+\tau)]\cos\left[\omega_n(t+\tau)+\theta\right]\sin(\omega_n t+\theta)d\tau \quad . \qquad (A39)$$

It will be noted that

$$\cos[\omega_n(t+\tau)+\theta]\cos(\omega_n t+\theta) = \tfrac{1}{2}\{\cos[2(\omega_n t+\theta)+\omega_n\tau] + \cos\omega_n\tau\} \qquad (A40)$$

and

$$\cos[\omega_n(t+\tau)+\theta]\sin(\omega_n t+\theta) = \tfrac{1}{2}\{\sin[2\omega_n(t+\tau)+\theta] - \sin\omega_n\tau\} \ . \qquad (A41)$$

Substituting the non-oscillatory terms of expressions (A40) and (A41) respectively in equations (A38) and (A39) yields [56]

$$E(\eta_1) = \frac{1}{4m\omega_n a}\int_{-\infty}^{\infty} E[w(t+\tau)w(t)]\cos\omega_n\tau\, d\tau \qquad (A42)$$

and

$$E(\xi_1) = \frac{1}{4m\omega_n a}\int_{-\infty}^{\infty} E[w(t+\tau)w(t)]\sin\omega_n\tau\, d\tau \ . \qquad (A43)$$

Since the process w(t) is stationary, the autocorrelation function is even [4]. Hence, equations (A42) and (A43) can be rewritten as

$$E(\eta_1) = \frac{1}{2m\omega_n a}\int_{0}^{\infty} E[w(t+\tau)w(t)]\cos\omega_n\tau\, d\tau \qquad (A44)$$

and

$$E(\xi_1) = 0 \ . \qquad (A45)$$

In terms of the spectral density $S(\omega)$, equation (A44) can be expressed as [4]

$$E(\eta_1) = \frac{\pi S(\omega_n)}{2m\omega_n a} \quad . \tag{A46}$$

Next, the autocorrelation functions of the zero mean processes

$$\eta = \eta_1 - E(\eta_1) \tag{A47}$$

and

$$\xi = \xi_1 - E(\xi_1) \tag{A48}$$

are determined. Since in each of equations (A28) and (A29) the presence of the second term on the right affects only the mean values, equations (A47) and (A48) can be written as

$$\eta \approx -w(t) \sin (\omega_n t + \theta_{-\Delta t}) \tag{A49}$$

and

$$\xi \approx -w(t) \cos (\omega_n t + \theta_{-\Delta t}) \quad . \tag{A50}$$

According to the preceding analysis, the random excitation $w(t)$ acts as if it were uncorrelated with $\theta_{-\Delta t}$; therefore, the actual autocorrelation function can be replaced by one shaped like a delta function [56]. That is

$$E[\eta(t)\eta(t+\tau)] \rightarrow \delta(\tau) \int_{-\infty}^{\infty} E[\eta(t)\eta(t+\tau)]\, d\tau \quad . \tag{A51}$$

The intensity coefficient is chosen to be $E[\eta(t)\eta(t+\tau)]$ so that integration with respect to τ of both sides of equation (A51) gives the same result. Taking account of equation (A49) it is found that

$$\int_{-\infty}^{\infty} E[\eta(t)\eta(t+\tau)]dt = \int_{-\infty}^{\infty} E[w(t)w(t+\tau)] \sin(\omega_n t+\theta) \sin[\omega_n(t+\tau)+\theta]d\tau$$

$$= \tfrac{1}{2} \int_{-\infty}^{\infty} E[w(t)w(t+\tau)] \cos \omega_n \tau d\tau$$

$$+ \text{ oscillatory terms} . \qquad \text{(A52)}$$

Neglecting the oscillatory terms, equation (A52) can be rewritten in terms of the spectral density $S(\omega_n)$ as

$$\int_{-\infty}^{\infty} E[\eta(t)\eta(t+\tau)]\, d\tau = \pi S(\omega_n) . \qquad \text{(A53)}$$

Similarly, the expression $E[\xi(t)\xi(t+\tau)]$ can be replaced as

$$E[\xi(t)\xi(t+\tau)] \rightarrow \delta(\tau) \int_{-\infty}^{\infty} E[\xi(t)\xi(t+\tau)]\, d\tau . \qquad \text{(A54)}$$

Using equation (A50) yields

$$\int_{-\infty}^{\infty} E[\xi(t)\xi(t+\tau)] = \int_{-\infty}^{\infty} E[w(t)w(t+\tau)] \cos(\omega_n t+\theta) \cos[\omega_n(t+\tau)+\theta]d\tau$$

$$= \tfrac{1}{2} \int_{-\infty}^{\infty} E[w(t)w(t+\tau)] \cos(\omega_n \tau)d\tau$$

$$+ \text{ oscillatory terms} . \qquad \text{(A55)}$$

Neglecting the oscillatory terms, equation (A55) can be rewritten as

$$\int_{-\infty}^{\infty} E[\xi(t)\xi(t+\tau)]\,d\tau = \pi S(\omega_n) \quad . \tag{A56}$$

According to the preceding analysis, equations (A19) and (A20) can be simplified as

$$\dot{a} = -\frac{c}{2m}a + \frac{\pi S(\omega_n)}{2am^2\omega_n^2} + \frac{[\pi S(\omega_n)]^{\frac{1}{2}}}{m\omega_n}\eta_2(t) \tag{A57}$$

and

$$\dot{\theta} = \frac{[\pi S(\omega_n)]^{\frac{1}{2}}}{am\omega_n}\xi_2(t) \tag{A58}$$

where

$$E[\eta_2(t)] = 0 \tag{A59}$$

$$E[\eta_2(t)\eta_2(t+\tau)] = \delta(\tau) \tag{A60}$$

$$E[\xi_2(t)] = 0 \tag{A61}$$

$$E[\xi_2(t)\xi_2(t+\tau)] = \delta(\tau) \quad . \tag{A62}$$

It is interesting to notice that if the zero mean value process $\eta_2(t)$ is neglected in (A57), the remaining part simply expresses the power balance of the system. Specifically, the rate of change of the energy per unit mass of the system $\omega_n^2 a\dot{a}$, equals the balance of the average power per unit mass $-(c/2m)a^2\omega_n^2$ dissipated through the damping mechanism plus the average power per unit mass [58] $\pi S(\omega_n)/m2$ imparted to the system by the excitation w(t).

The most important feature of the preceding simplification procedure is that equation (A57) governing the amplitude a(t) becomes uncoupled with θ.

Appendix B

Some Properties of a Class of Modified Laguerre Polynomials

The eigenvalues λ_n and the eigenfunctions A_n of the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left[\mu\left(a - \frac{1}{a}\right)\right] + \mu \frac{\partial^2 p}{\partial a^2} \; ; \qquad \mu > 0 \tag{B1}$$

are given by [52]

$$\lambda_n = 2n\mu \; ; \qquad n = 0, 1, \ldots \tag{B2}$$

and

$$A_n = \frac{1}{n!} a e^{-a^2/2} L_n\left(\frac{a^2}{2}\right) \; ; \qquad n = 0, 1, \ldots \tag{B3}$$

where

$$L_n\left(\frac{a^2}{2}\right) = e^{-a^2/2} \frac{d^n}{d\left(\frac{a^2}{2}\right)^n}\left[\left(\frac{a^2}{2}\right)^n e^{-a^2/2}\right] \; ; \qquad n = 0, 1, \ldots \tag{B4}$$

is the Laguerre polynomial of order n. When n = 0

$$A_0 = a e^{-a^2/2} \; . \tag{B5}$$

Therefore, relation (B3) can be rewritten as

$$A_n = \frac{1}{n!} A_0 L_n\left(\frac{a^2}{2}\right) \; ; \qquad n = 0, 1, \ldots \; . \tag{B6}$$

According to reference [59]

$$\int_0^\infty e^{-a^2/2} L_n\left(\frac{a^2}{2}\right) L_m\left(\frac{a^2}{2}\right) d\left(\frac{a^2}{2}\right) = n!\, m!\, \delta_{mn} \quad , \tag{B7}$$

where δ_{mn} is the Knonecker delta symbol. Using relations (B5) and (B6), equation (B7) can be rewritten as

$$\int_0^\infty \frac{A_m A_n}{A_0}\, da = \delta_{mn} \quad . \tag{B8}$$

Applying relation (B8) for $m = 0$ and $n \neq 0$ gives

$$\int_0^\infty A_m da = 0 \quad ; \qquad m = 1, \ldots \quad . \tag{B9}$$

According to reference [59]

$$L_{n+1}\left(\frac{a^2}{2}\right) = \left(2n+1 - \frac{a^2}{2}\right) L_n\left(\frac{a^2}{2}\right) - nL_{n-1}\left(\frac{a^2}{2}\right) \quad ; \qquad n = 0, 1, \ldots \tag{B10}$$

Because of relation (B6), equation (B10) implies that

$$(n+1)A_{n+1} = \left(2n+1 - \frac{a^2}{2}\right) A_n - nA_{n-1} \quad ; \qquad n = 0, 1, \ldots \quad . \tag{B11}$$

Equation (B11) is clearly very useful if numerical calculation of eigenfunctions A_n is desired. Multiplying both sides of equation (B11) by A_k/A_0 and integrating yields

$$(n+1)\int_0^\infty \frac{A_{n+1}A_k}{A_0}\,da = (2n+1)\int_0^\infty \frac{A_nA_k}{A_0}\,da - \tfrac{1}{2}\int_0^\infty \frac{a^2A_nA_k}{A_0}\,da$$

$$- n\int_0^\infty \frac{A_{n-1}A_k}{A_0}\,da \quad . \tag{B12}$$

By virtue of the orthonormality relation (B8), equation (B12) for $k = n$ implies

$$\int_0^\infty \frac{a^2A_n^2}{A_0}\,da = 2(2n+1) \quad ; \qquad n = 0, 1, \ldots \tag{B13}$$

for $k = n+1$ implies

$$\int_0^\infty \frac{a^2A_nA_{n+1}}{A_0}\,da = -2(n+1) \quad ; \qquad n = 0, 1, \ldots \tag{B14}$$

and for $k \neq n+1, n-1$ implies

$$\int_0^\infty \frac{a^2A_nA_k}{A_0}\,da = 0 \quad ; \qquad k \neq n+1, n, n-1 \; ; \; n = 0, 1, \ldots \quad . \tag{B15}$$

Multiplying both sides of relation (B11) by a^2A_k/A_0 and integrating yields

$$(n+1)\int_0^\infty \frac{a^2 A_{n+1} A_k}{A_0}\,da = (2n+1)\int_0^\infty \frac{a^2 A_n A_k}{A_0}\,da - \tfrac{1}{2}\int_0^\infty \frac{a^4 A_n A_k}{A_0}\,da - n\int_0^\infty \frac{a^2 A_{n-1} A_k}{A_0}\,da \quad . \qquad (B16)$$

Applying relation (B16) for k = n and using relations (B13) and (B14) gives

$$-2(n+1)^2 = 2(2n+1)^2 - \tfrac{1}{2}\int_0^\infty \frac{a^4 A_n^2}{A_0} + 2n^2 \quad . \qquad (B17)$$

After calculational manipulations equation (B17) can be rewritten as

$$\int_0^\infty \frac{a^4 A_n^2}{A_0}\,da = 8(3n^2 + 3n+1) \; ; \qquad n = 0, 1, \ldots \quad . \qquad (B18)$$

Applying relation (B16) for k = n+1 and using relations (B13), (B14) and (B15) yields

$$2(n+1)(2n+3) = -2(2n+1)(n+1) - \tfrac{1}{2}\int_0^\infty \frac{a^4 A_n A_{n+1}}{A_0}\,da \quad . \qquad (B19)$$

After calculational manipulations equation (B19) can be rewritten as

$$\int_0^\infty \frac{a^4 A_n A_{n+1}}{A_0}\, da = -16(n+1)^2 \quad ; \qquad n = 0, 1, \ldots \quad . \tag{B20}$$

Applying relation (B16) for $k = n+2$ and using relations (B14) and (B15) gives

$$-2(n+2)(n+1) = -\tfrac{1}{2} \int_0^\infty \frac{a^4 A_n A_{n+2}}{A_0}\, da \quad . \tag{B21}$$

After calculational manipulations equation (B21) can be rewritten as

$$\int_0^\infty \frac{a^4 A_n A_{n+2}}{A_0}\, da = 4(n+2)(n+1) \quad ; \qquad n = 0, 1, \ldots \quad . \tag{B22}$$

Applying relation (B16) for $k \neq n+2, n+1, n, n-1, n-2$ and using equation (B15) gives

$$\int_0^\infty \frac{a^4 A_n A_k}{A_0}\, da = 0 \quad ; \qquad n = 0, 1, \ldots \tag{B23}$$

where

$$k \neq n+2, n+1, n, n-1, n-2 \quad .$$

Clearly, the above procedure could be followed for the calculation of any integral of the form

$$I_{s,k,n} = \int_0^\infty \frac{a^{2s} A_n A_k}{A_0}\, da \quad ; \qquad s = 0, 1, \ldots \quad . \tag{B24}$$

For the solution of the example problems which are included in this thesis it was necessary to calculate $I_{s,k,n}$ for $s = 0$ and $s = 1$.

Next, the derivative dA_n/da will be expressed in terms of A_{n+1} and A_n. Differentiating equation (B3) gives

$$\frac{dA_n}{da} = \frac{1}{n!}\left[ae^{-a^2/2}\frac{dL_n}{da} + e^{-a^2/2}(1-a^2)L_n\right] . \tag{B25}$$

Using equation (B3), equation (B25) can be rewritten as

$$\frac{dA_n}{da} = \frac{A_n}{a} - aA_n + \frac{1}{n!}ae^{-a^2/2}\frac{dL_n}{da} . \tag{B26}$$

According to reference [60]

$$\frac{1}{n!}\frac{a^2}{2}\frac{dL_n}{d\left(\frac{a^2}{2}\right)} = n\left[\frac{L_n}{n!} - \frac{L_{n-1}}{(n-1)!}\right] . \tag{B27}$$

Clearly,

$$\frac{dL_n}{da} = a\frac{dL_n}{d\left(\frac{a^2}{2}\right)} . \tag{B28}$$

Therefore equation (B27) can be rewritten as

$$\frac{1}{n!}\frac{dL_n}{da} = \frac{2n}{a}\left[\frac{L_n}{n!} - \frac{L_{n-1}}{(n-1)!}\right] . \tag{B29}$$

By virtue of equations (B3) and (B29), relation (B26) can be put in the form

$$\frac{dA_n}{da} = \frac{A_n}{a} - aA_n + \frac{2n}{a}(A_n - A_{n-1}) \quad . \tag{B30}$$

Using the recursive formula, equation (B11), the last equation can be rewritten as

$$\frac{dA_n}{da} = \frac{1}{a}\left[2(n+1)A_{n+1} - (2n+1)A_n\right] \; ; \qquad n = 0, 1, \ldots \quad . \tag{B31}$$

Another useful form of equation (B11) is

$$(n+1)(A_{n+1} - A_n) = n(A_n - A_{n-1}) - \frac{a^2}{2} A_n \quad . \tag{B32}$$

If relation (B32) is used repeatedly the following formula is obtained

$$2n(A_n - A_{n-1}) = -a^2(A_{n-1} + \cdots + A_0) \quad . \tag{B33}$$

This formula could be used to study the behavior of $\sum_{j=0}^{r} A_j$ as $r \to \infty$.

Next, consider the integral

$$I_{n,1} = \int_0^\infty aA_n da \quad . \tag{B34}$$

This integral may also be expressed (B34) as

$$I_{n,1} = \int_0^\infty aA_n da = \int_0^\infty A_n \frac{d\left(\frac{a^2}{2}\right)}{da} da$$

$$= \frac{a^2}{2} A_n \Big|_0^\infty - \tfrac{1}{2} \int_0^\infty a^2 \frac{dA_n}{da} \quad . \tag{B35}$$

Clearly,

$$\frac{a^2}{2} A_n \Big|_0^\infty = \frac{a^3}{2n!} e^{-a^2/2} L_n\left(\frac{a^2}{2}\right) \Big|_0^\infty = 0 \quad . \tag{B36}$$

Hence, equation (B35) can be rewritten as

$$I_{n,1} = -\tfrac{1}{2} \int_0^\infty a^2 \frac{dA_n}{da} \quad . \tag{B37}$$

Upon using formula (B31) for the derivative dA_n/da in equation (B37) gives

$$I_{n+1,1} = \frac{n-\frac{1}{2}}{n+1} I_{n,1} \qquad n = 0, 1, \ldots \quad . \tag{B38}$$

Relation (B38) can be used for recursive calculation of I_n. For $n = 0$ it is easily shown that

$$I_{0,1} = \int_0^\infty a^2 e^{-a^2/2} \, da = \sqrt{\frac{\pi}{2}} \quad . \tag{B39}$$

It can easily be proved that

$$\lim_{n \to \infty} I_{n,1} = 0 \quad . \tag{B40}$$

Attention is now turned to the integral

$$I_{n,2} = \int_0^\infty a^2 A_n da \quad . \tag{B41}$$

Integrating both sides of equation (B11) gives

$$(n+1)\int_0^\infty A_{n+1}da = (2n+1)\int_0^\infty A_n da - \tfrac{1}{2}\int_0^\infty A_n a^2 da$$

$$- n\int_0^\infty A_{n-1}da \quad . \tag{B42}$$

Because of (B9), equation (B42) implies that

$$I_{n,2} = 0 \quad ; \qquad n = 2,3,\ldots \tag{B43}$$

$$I_{1,2} = 2 \tag{B44}$$

and

$$I_{0,2} = 2 \quad . \tag{B45}$$

It is clear from the preceding analysis that the integral

$$I_{n,s} = \int_0^\infty a^s A_n da \quad ; \qquad s = 3,\ldots \tag{B46}$$

may be calculated by the same techniques used for the calculations of the integrals $I_{n,1}$ and $I_{n,2}$.